# JANUARY 2014 ANALYSIS QUALIFYING EXAM

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## 1. Problem 1

Let $\epsilon > 0$. Give $\mathbb{R}_{\geqslant 0}$ the induced topology and consider the open sets

$$f_n^{-1}([0,\epsilon))$$

Note that these sets are open by continuity. Since $f_n \to 0$ pointwise, we have that

$$\bigcup_{n\geqslant 1} f_n^{-1}([0,\epsilon))$$

is an open cover of $E$. Since $E$ is compact, we may select a finite subcover $\{f_{n_1}^{-1}([0,\epsilon)),\dots,f_{n_k}^{-1}([0,\epsilon))\}$.

Observe now that since the $f_n$ are monotone decreasing, $f_n^{-1}([0,\epsilon)) \subset f_{n+1}^{-1}([0,\epsilon))$, in which case we see that

$$f_{n_k}^{-1}([0,\epsilon)) = E$$

And, for all $\ell \geqslant n_k$,

$$f_\ell^{-1}([0,\epsilon)) = E$$

That is, for all $\ell > n_k$, $f_\ell(x) < \epsilon$ for every $x \in E$, so that $f_n$ converges uniformly to 0.

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## 2. Problem 2

Observe first that

$$\left|e^{z^2}\right| = e^{\mathrm{Re}(z^2)}$$

Then the problem says that

$$\frac{f(z)}{e^{z^2}}$$

is bounded. Since $f(z)$ and $e^{z^2}$ are both entire, Liouville's theorem gives that the above quotient must be a constant. That is,

$$\frac{f(z)}{e^{z^2}} = a$$

So that $f(z) = ae^{z^2}$.

## 3. Problem 3

We see that the only singularity in the contour $|z-1| = 1$ is at $z = 1$. By Cauchy's Residue formula,

$$\int_{|z-1|=1} \frac{e^z}{z^4-1}dz = 2\pi i \cdot \lim_{z\to 1} \frac{e^z}{1+z+z^2+z^3} = \frac{e\pi i}{2}$$

## 4. Problem 4

Define $g(x) := f(x) - (x-1)$. By the problem's assumptions,

$$\int_0^1 g(x)x^n dx = 0$$

for all $n$. We then see (by linearity of integration) that for all $\ell \in \mathbb{R}[x]$,

$$\int_0^1 g(x)\ell(x)dx = 0$$

Let $\epsilon > 0$. By the Stone-Weierstrass theorem, we may find $p \in \mathbb{R}[x]$ such that

$$||g-p||_1 < \frac{\epsilon}{1+||g||_\infty}$$

Note that $||g||_\infty < \infty$ since $g$ is continuous on a compact set. We then see:

$$\begin{aligned}\int_0^1 g^2(x)dx &= \int_0^1 g(x)\big(g(x) - p(x)\big)dx \\ &\leqslant ||g||_\infty ||g-p||_1 \quad \text{(Hölder's)} \\ &< \epsilon\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we deduce that $\int_0^1 g^2(x)dx = 0$. Since $g$ is continuous, this is possible if and only if $g \equiv 0$; that is, $f(x) = 1 - x$, so that $1 - x$ is the only continuous function on $[0, 1]$ satisfying

$$\int_0^1 f(x)x^n dx = \frac{1}{(n+1)(n+2)}$$

for all $n$.

## 5. Problem 5

Define $A := \{x : \lambda(E_x) \geqslant 1/2\}$. Then, $\lambda(A) \geqslant 3/4$ by assumption; we see:

$$\begin{aligned}\lambda \times \lambda(E) &= \int_E d\lambda \times \lambda(x, y) \\ &= \int_{[0,1]} \int_{E_x} d\lambda(y) d\lambda(x) \\ &\geqslant \int_A \int_{E_x} d\lambda(y) d\lambda(x) \\ &\geqslant \frac{1}{2} \int_A d\lambda(x) \\ &\geqslant \frac{1}{2}\frac{3}{4} = \frac{3}{8}\end{aligned}$$

## 6. Problem 6

Define $g_n := \inf_{k \geqslant n} f_n$, where $f_n$ is our sequence of functions. Obviously $g_n \leqslant f_n$, so that

$$\int_E g_n \leqslant \int_E f_n$$

Since this in fact holds for all $n$, we have the stronger inequality:

$$\int_E g_n \leqslant \inf_{k\geqslant n} \int_E f_k$$

Note that $g_n$ is an increasing sequence of functions. By Lebesgue's monotone convergence theorem,

$$\lim_{n\to\infty} \int_E g_n = \int_E \lim_{n\to\infty} g_n$$

Taking the limit in our inequality then yields:

$$\int_E \liminf_{n\to\infty} f_n \leqslant \liminf_{n\to\infty} \int_E f_n$$

And Fatou's Lemma is proved.

## 7. Problem 7

Let $\epsilon > 0$. We may find a smooth $f_c \in L_1$ with compact support such that $||f - f_c||_1 \leqslant \epsilon/2$ by density of smooth, compactly supported functions.

Now, consider the difference $||f_n - f_c(1-1/n)||_1$. Making the change of variable $z = x - 1/n$ in this difference, we see

$$||f_n - f_c(1 - 1/n)||_p = ||f - f_c||_p < \epsilon/2$$

Now consider $||f_c(x-1/n)-f_c||_p$ and suppose that Supp $f_c \subset [-M, M]$. As $f_c$ is smooth, in particular it is continuous on a compact interval, hence bounded. Then, note that $|f_c(x - 1/n) - f_c(x)| \leqslant 2||f||_\infty$. By the dominated convergence theorem and continuity,

$$\begin{aligned}\lim_{n\to\infty} \int_{-M}^{M} |f_c(x - 1/n) - f_c(x)|dx &= \int_{-M}^{M} \lim_{n\to\infty} |f_c(x - 1/n) - f_c(x)|dx \\ &= \int_{-M}^{M} |f_c(x) - f_c(x)|dx \\ &= 0\end{aligned}$$

Whence

$$\begin{aligned}\lim_{n\to\infty} ||f(x-1/n)-f||_1 \leqslant \lim_{n\to\infty} ||f-f_c||_1 + \\ ||f_c(x-1/n)-f_c(x)||_1 + ||f_c(x-1/n)-f(x-1/n)||_1 \\ < \epsilon/2 + \lim_{n\to\infty} ||f_c(x-1/n)-f_c(x)||_1 + \epsilon/2 \\ = \epsilon\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we see that

$$\lim_{n\to\infty} ||f(x-1/n)-f(x)||_1 = 0$$

as desired.

## 8. Problem 8

Let $\epsilon > 0$. Since $f$ is integrable on $\mathbb{R}$, we may find $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus[-N,N]} f d\lambda < \epsilon$$

Then, since $[-N,N]$ is closed an compact, by assumption $\int_{[-N,N]} f d\lambda = 0$; then

$$\int_{\mathbb{R}} f d\lambda = \int \mathbb{R}\setminus[-N,N] f d\lambda + \int_{[-N,N]} f d\lambda$$
$$< \epsilon$$

As $\epsilon > 0$ is arbitrary, we conclude that $\int_{\mathbb{R}} f d\lambda = 0$, as desired.

## 9. Problem 9

Recall first that if we may write $f(x) - f(0) = \int_0^x f' d\mu$, then $f$ is absolutely continuous. We also have the standard inequality that holds in general:

$$\int_a^b f' d\mu \leqslant f(b) - f(a)$$

Now we may proceed with the proof. By Lebegue's theorem on monotone functions, we have that $f'$ exists almost everywhere. Let $x \in (0,1)$:

$$\begin{aligned} 0 &= f(1) - f(0) - \int_0^1 f' d\mu \\ &= f(1) - f(x) - \int_x^1 f' d\mu + f(x) - f(0) - \int_0^x f' d\mu \end{aligned}$$

By the above, however,

$$f(1) - f(x) - \int_x^1 f' d\mu \geqslant 0, \quad f(x) - f(0) - \int_0^x f' d\mu \geqslant 0$$

In which case we have the sum of two nonnegative functions being equal to 0; this is only psosible if both functions are themselves identically 0. Thus we deduce

$$f(x) - f(0) = \int_0^x f' d\mu$$

And $f$ is absolutely continuous, as contended.

## 10. Problem 10

*(a).* False. Let $\epsilon < 1$ and set

$$E := \bigcup_{q \in \mathbb{Q}} B_{\epsilon/2^{n+1}}(q)$$

This is open as the union of open sets, in which case we deduce that $E^c$ is a closed subset of $[0,1]$, and is hence compact. Also, by construction, $\mathbb{Q} \subset E$ and $\mu(E) \leqslant \epsilon$, in which case $E^c \subset \mathbb{Q}^c$, and, $\mu(E^c) \geqslant 1 - \epsilon > 0$.

Thus we have constructed a compact subset of the irrationals with strictly positive measure, in which case the statement must be false.

*(b).* False. Set $f_n(x) := \sin(2n^2 x) dx$. By construction,

$$||f_n||_{L^1} \leqslant \frac{1}{n^2}$$

However, $\lim_{n\to\infty} f_n$ does not exist, in which case $f_n \not\to 0$.

*(c).* True. Set $f(z) := 0$. Now, assuming the problem wanted a non-constant function, the statement is still true. Set $f(z) := \sin(\pi z)$.

*(d).* False. Since $f$ is Lipschitz, it is also absolutely continuous. We may then write

$$f(x) = f(0) + \int_0^x f' d\mu$$

Since $f' = 0$ a.e, we find $f(x) = f(0)$ for all $x$; that is, $f$ is constant.

*(e).* False. By the Schwarz Lemma, if $|f(z)| \leqslant 3$, we must also have $|f'(0)| \leqslant 3$.