JANUARY 2008 ANALYSIS QUALIFYING EXAM

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1. Problem 1

Note that $k^k > k!$ for all $k > 0$, so that by the comparison test

$$\sum_{k=!}^{\infty} k^{-k} \leqslant \sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$$

In which case s_n is convergent, hence Cauchy.

2. Problem 2

Let $x_0 \in X$ and define x_n inductively by $x_n = \Omega(x_{n-1})$. Then, we can show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy as for $m > n$,

$$\begin{aligned}\rho(x_n, x_m) &\leqslant \rho(x_m, x_{m-1}) + \cdots + \rho(x_{n+1}, x_n)\\ &\leqslant \big(\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n\big)\rho(x_1, x_0)\\ &= \frac{\lambda^n - \lambda^m}{1-\lambda}\rho(x_1, x_0) \to 0 \text{ as } m,\ n \to \infty\end{aligned}$$

By completeness of X, we deduce that $x_n \to x \in X$. Now, consider $\Omega(x)$; we want to show that x must be a fixed point:

$$\rho(\Omega(x), x) \leqslant \rho(x, x_{n+1}) + \rho(x_{n+1}, \Omega(x))$$
$$\leqslant \rho(x, x_{n+1}) + \lambda\rho(x_n, x)$$

Letting $n \to \infty$ on the right, this must tend to 0, in which case

$$\rho(x, \Omega(x)) = 0$$

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That is, $\Omega(x) = x$. Lastly, it remains to show uniqueness. Suppose then that x and y are two fixed points of Ω; then:

$$\rho(x,y) = \rho(\Omega(x),\Omega(y)) \leqslant \lambda\rho(x,y)$$

Since $\lambda < 1$, we must have $\rho(x,y) = 0$, so that $x = y$.

3. Problem 3

Yes, this is uniform. Let $\epsilon > 0$; since $\frac{x^n}{e^x} \to 0$ as $x \to \infty$, there exists $M \in \mathbb{R}_+$ such that $\frac{x^n}{e^x} < \epsilon$ for all $x > M$. As $\frac{x^n}{(2n)!} < \frac{x^m}{(2m)!}$ for all $n > m$, we see that for all $m > n$, $x > M$,

$$\frac{x^m}{(2m)!} < \frac{x^n}{(2n)!} < \frac{\epsilon}{(2n)!} < \epsilon$$

Similarly, when $x < M$, we have that $\frac{x^n}{(2n)!} < \frac{M^n}{(2n)!} \to 0$, so that we may find $N \in \mathbb{N}$ such that for all $n > N$ and $x < M$,

$$\frac{x^n e^{-x}}{(2n)!} \leqslant \frac{x^n}{(2n)!} < \epsilon$$

So that, choosing $n > N$,

$$\frac{x^n e^{-x}}{(2n)!} < \epsilon$$

whence $\frac{x^n e^{-x}}{(2n)!} \to 0$ uniformly on $[0, \infty)$.

4. Problem 4

Note that γ only encloses the pole of order 1 at the point $z = -1$. Computing residues,

$$\begin{aligned}\text{Res}\Big(\frac{1}{1+z^3}, -1\Big) &= \lim_{z\to -1} \frac{1+z}{1+z^3} \\ &= \lim_{z\to -1} \frac{1}{1-z+z^2} = \frac{1}{3}\end{aligned}$$

By Cauchy's residue theorem, we see

$$\int_\gamma \frac{1}{1+z^3} dz = \frac{2\pi i}{3}$$

5. Problem 5

Observe that

$$\begin{aligned}\int_a^b f(x)^2dx &= \int_a^b f(x)^{2/3}\cdot f(x)^{1/3}dx\\ &\leqslant \Big(\int_a^b f(x)dx\Big)^{2/3}\Big(\int_a^b f(x)^4dx\Big)^{2/3}\end{aligned}$$

Taking square roots of the above, we see

$$||f||_2 \leqslant ||f||_1^{1/3}\cdot ||f||_4^{2/3}$$

As contended.

6. Problem 6

Note first that since $|f_n|\leqslant g$ for all n, letting $n\to\infty$ gives $|f|\leqslant |g|$ as well. By Fatou's lemma, we see

$$\begin{aligned}0 &\leqslant \int_E 2^p - \lim_{n\to\infty}|f_n-f|^p d\mu\\ &\leqslant \liminf_{n\to\infty}\Big(\int_E 2^p g - \int_E |f_n-f|d\mu\Big)\\ &= \int_X 2^p g - \limsup_{n\to\infty}\int_E |f_n-f|^p d\mu\\ \implies &\limsup_{n\to\infty}\int_E |f_n-f|^p d\mu \leqslant 0\end{aligned}$$

So,

$$||f_n-f||_p\to 0$$

and, by the triangle inequality we see $||f_n||_p\to||f||_p$, as desired.

7. Problem 7

Consider

$$f(x) := \int_0^x \chi_A(x)dx$$

This function is absolutely continuous by absolute continuity of integration, and, in particular, by the intermediate value property of continuous functions, there exists $y \in [0,1]$ such that $f(y) = b$. That is, $\mu(A \cap [0,y]) = b$.

Obviously $A \cap [0,y]$ is measurable; it remains to see that there exists a closed set $B \subset A$ with $\mu(B) = A$. Choose $\epsilon = a - b$. By definition of Lebesgue measure we may find a closed set F with $A \cap [0,y] \subset F \subset A$ such that $\mu(A \backslash F) \leqslant \epsilon$. However, this implies

$$\mu(A \backslash (A \cap [0,y])) = a - b \leqslant \mu(A \backslash F) \leqslant a - b$$

In which case $\mu(A \backslash F) = a - b$, so that $\mu(F) = b$, as desired.

8. Problem 8

Note that $f_n \chi_E \leqslant \sup_n f_n \chi_E \leqslant \sup_n f_n \in L^1(\mathbb{R})$, where $\sup_n f_n \in L^1(\mathbb{R})$ by assumption. By Lebesgue's dominated convergence theorem,

$$\int_E f_n d\mu = \int f_n \chi_E d\mu \to \int f \chi_E d\mu = \int_E f d\mu$$

as desired.

9. Problem 9

Since $[a,b]$ is compact, $m := \inf_{x \in [a,b]} |f(x)| > 0$. As f has bounded total variation, we know

$$\sup_{P \text{ partition}} \sum_{k=1}^N |f(b_k) - f(a_k)| < \infty$$

so that

$$\sup_{P \text{ partition}} \sum_{k=1}^N \left| \frac{1}{f(b_k)} - \frac{1}{f(a_k)} \right| \leqslant \frac{1}{m^2} \sup_{P \text{ partition}} \sum_{k=1}^N |f(b_k) - f(a_k)| \\ < \infty$$

So that $1/f$ also has bounded variation.

10. Problem 10

By absolute continuity,

$$|f(b) - f(a)| = \Big| \int_a^b f'(t)dt \Big|$$

so we compute:

$$\begin{aligned} |f(b) - f(a)| &= \Big| \int_a^b f'(t)dt \Big| \\ &\leqslant \int_a^b |f'(t)|dt \\ &\leqslant ||f'||_p |b-a|^{1-1/p} \quad \text{(Hölder's)} \end{aligned}$$

So that $C = ||f'||_p$.