FALL 2017 QUALIFYING EXAM

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1. Problem 1

Our contour γ is a circle of radius 5; we see that the singularities of the integrand are 0 and π, so these all lie within our contour. By Cauchy's Residue formula,

$$\int_\gamma ze^{3/z} + \frac{\cos(z)}{z^2(z-\pi)^3} dz = 2\pi i \sum \text{Res}(f(z), z_0)$$

Where the above sum is taken over our residues. We first compute the residue of $ze^{3/z}$:

$$\begin{aligned} ze^{3/z} &= z\Big(\sum_{n=0}^{\infty} \frac{3^n}{n!z^n}\Big) \\ &= \sum_{n=0}^{\infty} \frac{3^n}{n!z^{n-1}} \end{aligned}$$

Then the coefficient of z^{-1} in the above is our residue, and one immediately sees that this is 9/2.

Similarly, for the second term in our summand, we can find the residue at 0:

$$\begin{aligned} &\lim_{z\to 0} \frac{d}{dz}\Big(\frac{\cos(z)}{(z-\pi)^3}\Big) \\ &\qquad = \lim_{z\to 0} \frac{-\sin(z)}{(z-\pi)^3} - \frac{3\cos(z)}{(z-\pi)^4} \\ &\qquad = -\frac{3}{\pi^4} \end{aligned}$$

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And, the residue at π gives

$$\begin{aligned}\frac{1}{2}\lim_{z\to\pi}\frac{d^2}{dz^2}\Big(\frac{\cos(z)}{z^2}\Big) &= \frac{1}{2}\lim_{z\to 0}\Big(\frac{-\cos(z)}{z^2}+\frac{4\sin(z)}{z^3}+\frac{6\cos(z)}{z^4}\Big)\\ &= \frac{1}{2\pi^2}-\frac{3}{\pi^2}\end{aligned}$$

Using this with the integral formula, we get

$$\int_\gamma ze^{3/z}+\frac{\cos(z)}{z^2(z-\pi)^3}dz = \frac{i}{\pi}+9\pi i$$

2. Problem 2

Suppose $\text{Im}(f) \geqslant 0$ and consider the function

$$g(z) := e^{if(z)}$$

This is entire, since f is. We also see

$$|g(z)| = \left|e^{if(z)}\right| = e^{-\text{Im}f(z)}$$

And since $\text{Im}f(z) \geqslant 0$, $e^{-\text{Im}f(z)} \leqslant 1$. Thus g is bounded and entire, hence constant, which means that f must also be constant.

3. Problem 3

Let $x_n \to x$ be a sequence of points in $K+C$. We want to show that $x \in K+C$; note that $x_n = k_n + c_n$ for $c_n \in C$, $k_n \in K$.

By compactness, we may choose a convergent subsequence k_{n_j} such that $k_{n_j} \to k \in K$ as $j \to \infty$. Consider then the subsequence $c_{n_j} = x_{n_j} - k_{n_j}$; as $j \to \infty$, this converges to $x-k$, and since C is closed, $x - k \in C$.

We then note that $x = (x-k)+k$ is an element of the Minkowski sum $K+C$.

4. Problem 4

Define $f(x) := \frac{d(x,B)-d(x,A)}{d(x,A)+d(x,B)}$.

This is obviously continuous as the ratio of continuous functions. The bottom cannot vanish since the sets A and B are disjoint. More specifically, if $d(x,A)+d(x,B)=0$, then $x \in \overline{A}$ and $x \in \overline{B}$, since these sets are closed, $x \in A \cap B = \varnothing$, which is impossible.

Whenever $x \in A$, $d(x,A)=0$, so $f(x)=1$. Similarly, for $x \in B$, $f(x)=-1$. Also, by the triangle inequality and positive definiteness of our metric,

$$\big|d(x,B)-d(x,A)\big| \leqslant d(x,A)+d(x,B)$$

Which yields that $-1 \leqslant f \leqslant 1$.

5. Problem 5

Suppose first that f is measurable. Recall that the metric function $x \mapsto d(y,x)$ is continuous with respect to x, in which case the composition

$$x \mapsto f(x) \mapsto d(y,f(x))$$

is a measurable function as the composition of a continuous function with a measurable function.

To see that this holds, note that for every Borel set of $\mathbb{R}$, the inverse image under a continuous function remains a Borel set. Now suppose f is measurable and g is continuous. If U is any Borel set, then,

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)))$$

Since $g^{-1}(U)$ is Borel, the preimage under f is measurable, so that the composition is itself measurable.

Let us prove the converse now. First, recall that for a separable metric space, measurability of a given function is equivalent to showing measurability on all open balls.

To see this, choose any Borel set of the codomain. Then, by definition, this is either a countable union or intersection of open subsets. We may employ second countability to cover our Borel set with countably many open sets; thus if the inverse image of every open ball is measurable, f is measurable. The converse holds trivially.

Now, back to the problem at hand. We see:

$$\begin{aligned} g_y^{-1}\big((-\infty,\epsilon)\big) &= \{x \mid d(y,f(x)) < \epsilon\} \\ &= f^{-1}(B_\epsilon(y)) \end{aligned}$$

Since g_y is assumed measurable, we see that $f^{-1}(B_\epsilon(y))$ is also measurable. As $\epsilon > 0$ is arbitrary, f must be measurable of open ball. By separability, f must be measurable.

6. Problem 6

Let $\epsilon > 0$. We may find a smooth $f_c \in L_p$ with compact support such that $||f - f_c||_p \leqslant \epsilon/2$ by density of smooth, compactly supported functions.

Now, consider the difference $||\tau_y f - \tau_y f_c||_p$. Making the change of variable $z = x - y$ in this difference, we see

$$||\tau_y f - \tau_y f_c||_p = ||f - f_c||_p < \epsilon/2$$

Now consider $||\tau_y f_c - f_c||_p$ and suppose that f Supp $f_c \subset [-n,n]$. As f_c is smooth, in particular it is continuous on a compact interval, hence bounded. Then, note that $|f_c(x-y) - f_c(x)| \leqslant 2||f||_\infty$. By the dominated convergence theorem and continuity,

$$\begin{aligned}\lim_{y\to 0}\int_{-n}^{n}|f_c(x-y)-f_c(x)|^p dx &= \int_{-n}^{n}\lim_{y\to 0}|f_c(x-y)-f_c(x)|^p dx\\ &= \int_{-n}^{n}|f_c(x)-f_c(x)|^p dx\\ &= 0\end{aligned}$$

Whence

$$\begin{aligned}\lim_{y\to 0}||\tau_y f - f||_p &\leqslant \lim_{y\to 0}\Big(||f-f_c||_p + ||\tau_y f_c - f_c||_p + ||\tau_y f_c - \tau_y f||_p\Big)\\ &< \epsilon/2 + \lim_{y\to 0}||\tau_y f_c - f_c||_p + \epsilon/2\\ &= \epsilon\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we see that

$$\lim_{y\to 0}||\tau_y f - f||_p = 0$$

as desired.

7. Problem 7

Consider the space of exponential polynomials

$$\mathcal{S} := \{\sum_n a_n e^{nx} \mid a_n \in \mathbb{R}\}$$

$\mathcal{S}$ contains all constants, since we may just set $n = 0$; by the Stone-Weierstrass theorem, this is dense in $[0,1]$. By the condition on f, we also see that for any $\sum_n a_n e^{nx} \in \mathcal{S}$,

$$\int_0^1 f(x)\sum_n a_n e^{nx} dx = \sum_n a_n \int_0^1 f(x)e^{nx} dx = 0$$

Let $\epsilon > 0$. By density, we may find $p \in \mathcal{S}$ such that $||f-p||_1 < \epsilon/||f||_\infty$; consider now the quantity $\int_0^1 f^2(x)dx$:

$$\begin{aligned}\int_0^1 f^2(x)dx &= \int_0^1 f(x)(f(x)-p(x))dx\\ &\leqslant ||f||_\infty ||f-p||_1 \quad \text{(Hölder's)}\\ &< \epsilon\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we conclude that $\int_0^1 f^2(x)dx = 0$; since $f^2(x) \geqslant 0$, it must be that $f(x) \equiv 0$ almost everywhere, which completes the proof.

8. Problem 8

We will first need:

Theorem 8.1 (Egoroff's Theorem)**.** *Let (X, Σ) be a finite measure space and let $f_n \to f$ be a sequence of measurable functions converging pointwise a.e to a measurable f. Then f_n converges almost uniformly and in measure to f.*

Let $\epsilon \in (0, 1)$. We first proceed to show that $f \in L_p(X, \mu)$. By uniform integrability, we also have a uniform bound such that $\int_X |f_n|^p d\mu \leqslant M$ for all $n \in \mathbb{N}$. To see this, choose δ' such that $\int_E |f_n|^p d\mu < 1$ for all E with $\mu(E) < \delta'$. Since X is of finite measure, we may partition X into finitely many sets $\{E_k\}_{k=1}^M$ each with measure less than δ'. Then, for all n,

$$\int_X |f_n|^p d\mu \leqslant \sum_{k=1}^M \int_{E_k} |f_n|^p d\mu < M$$

Then, by Fatou's lemma,

$$\int_X |f|^p d\mu \leqslant \liminf_{n\to\infty} \int_X |f_n|^p \leqslant M$$

So that $f \in L_p(X, \mu)$. Absolute continuity of integration now guarantees the existence of δ_1 such that for all sets E with $\mu(E) < \delta_1$,

$$\int_E |f|^p d\mu < \frac{\epsilon}{3 \cdot 2^p}$$

Similarly, by uniform integrability we are guaranteed the existence of δ_2 such that for all E with $\mu(E) < \delta_2$,

$$\int_E |f_n|^p d\mu \leqslant \frac{\epsilon}{3 \cdot 2^p} \quad \text{for all } n \in \mathbb{N}$$

Set $\delta := \min\{\delta_1, \delta_2\}$. By Egoroff's theorem, there exists a set F with $\mu(F) < \delta$ and such that $f_n \to f$ uniformly on F^c. By uniform convergence, on F^c we may choose $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n - f| < \frac{\epsilon}{3 \cdot (1 + \mu(F^c))}$$

Now, putting this all together:

$$\begin{aligned}
\int_X |f_n - f|^p d\mu &= \int_{F^c} |f_n - f|^p d\mu + \int_F |f_n - f|^p d\mu \\
&\leqslant \int_{F^c} |f_n - f|^p d\mu + 2^p \Big(\int_F |f|^p d\mu + \int_F |f_n|^p d\mu \Big) \\
&< \frac{\epsilon^p}{3^p \cdot (1 + \mu(F^c))^p} \cdot \mu(F^c) + 2^p \Big(\frac{\epsilon}{3 \cdot 2^p} + \frac{\epsilon}{3 \cdot 2^p} \Big) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

Whence $||f_n - f||_p \to 0$, as desired.

9. Problem 9

(a). Consider $\mathbb{Z}$ and $\alpha\mathbb{Z}$ for any irrational α. Both of these sets are closed, however, $\mathbb{Z} + \alpha\mathbb{Z}$ is dense in $\mathbb{R}$, so this sum is obviously not closed.

(b). Let $f \in L_q$. Consider

$$E := \{x \mid |f(x)| > 1\}$$

and rewrite

$$f = f_1 + f_2$$

with $f_1 = f\chi_E$, $f_2 = f\chi_{E^c}$. Then,

$$\begin{aligned} ||f_1||_p &= \Big(\int_E |f|^p d\mu \Big)^{1/p} \\ &= \Big(\int_E |f|^{p-q}|f|^q d\mu \Big)^{1/p} \\ &\leqslant \Big(\int_E |f|^q d\mu \Big)^{1/p} \\ &\leqslant ||f_1||_q^{q/p} < \infty \end{aligned}$$

So $f_1 \in L_p$. Similarly,

$$\begin{aligned} ||f_2||_r &= \Big(\int_{E^c} |f|^r d\mu \Big)^{1/r} \\ &= \Big(\int_{E^c} |f|^{r-q}|f|^r d\mu \Big)^{1/r} \\ &\leqslant \Big(\int_{E^c} |f|^q d\mu \Big)^{1/r} \\ &\leqslant ||f_2||_q^{q/r} < \infty \end{aligned}$$

So $f_2 \in L_r$, from which we deduce that $f \in L_p + L_r \implies L_q \subset L_p + L_r$.

(c). Argue by contraposition. If E is not dense in $[0,1]$, we may find $x \in [0,1]$ and $\epsilon > 0$ such that $B_\epsilon(x) \cap E = \varnothing$. Then,

$$\begin{aligned} 1 - \epsilon &= \mu([0,1] \backslash B_\epsilon(x)) \\ &\geqslant \mu(E) \end{aligned}$$

So that $\mu(E) < 1$.

(d). Let $\epsilon > 0$. Given any $\delta > 0$, there is $N \in \mathbb{N}$ such that $||f_n - f||_p < \delta\epsilon^{1/p}$ for all $n > N$. By Chebyshev's inequality,

$$\mu(\{x \mid |f_n(x) - f(x)| \geqslant \delta\}) \leqslant \Big(\frac{||f_n - f||_p}{\delta} \Big)^p < \epsilon$$

So that $f_n \to f$ in measure.

(e). We find the harmonic complement by means of the Cauchy Riemann equations. The answer is

$$f(x,y) = xy - x + y + i\Big(\frac{y^2}{2} - \frac{x^2}{2} - y - x \Big)$$