FALL 2016 QUALIFYING EXAM

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1. Problem 1

Suppose the conclusion is false; then we may find A_n with $\text{diam}(A_n) < 1/n$, and $x_n \in A_n$ with $A_n \not\subset U$. Choose a convergent subsequence $x_{n_k} \to x \in K$ by compactness of K. Since U is open, we may find $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Since $x_{n_k} \to x$, we may choose k such that $d(x_{n_k}, x) < \epsilon/2$, and additionally, we may choose k large enough such that $\text{diam}(A_{n_k}) < \epsilon/2$. Taking k sufficiently large to satisfy the above, we see that for any $a \in A_{n_k}$,

$$\begin{aligned} d(x,a) &\leqslant d(x, x_{n_k}) + d(x_{n_k}, a) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

From which we deduce that $A_{n_k} \subset B_\epsilon(x) \subset U$, which is a contradiction.

2. Problem 2

Assume the conclusion is false. If $(E + 1/n) \cap (E + 1/m) = \varnothing$ for every $n, m \in \mathbb{N}$, then, consider

$$\bigcup_{n \geqslant 1} (E + 1/n)$$

Since E is bounded, this union is also bounded. We also see,

$$\mu\Big(\bigcup_{n\geqslant 1}(E+1/n)\Big) = \sum_{n\geqslant 1} \mu(E) = \infty$$

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which is impossible, so that there must exist n and m such that

$$(E + 1/n) \cap (E + 1/m) \neq \varnothing$$

Now, if the above holds, we may find x, $y \in E$ such that $x + 1/n = y + 1/n$, in which case

$$x - y = 1/m - 1/n \in \mathbb{Q}$$

As asserted.

3. Problem 3

Define $g_n := \max\{f_1, \dots, f_n\}$. Obviously $f_n \leqslant g_n$ for every n, and g_n is integrable for every n. By Lebesgue's dominated convergence theorem,

$$\int_E f_n d\mu \to 0$$

4. Problem 4

(a). Certainly $\chi_E \leqslant 1$ for any values of x and t, so

$$n \int_0^{1/n} \chi_E(x+t)dt \leqslant n \cdot \frac{1}{n} = 1$$

And obviously $0 \leqslant f_n$.

(b). After a change of variable, we see that

$$f_n(x) = n \int_x^{1/n+x} \chi_E(t)dt$$

We then see

$$\begin{aligned}|f_n(x+h)-f_n(x)| &= n\Big|\int_{x+h}^{1/n+x+h}\chi_E(t)dt-\int_x^{1/n+x}\chi_E(t)dt\Big|\\ &= n\Big|\int_x^{x+h}\chi_E(t)dt+\int_{x+1/n}^{1/n+x+h}\chi_E(t)dt\Big|\\ &\leqslant n\Big|\int_x^{x+h}\chi_E(t)dt\Big|+\Big|\int_{x+1/n}^{1/n+x+h}\chi_E(t)dt\Big|\\ &\leqslant 2n|h|\end{aligned}$$

So that f_n is Lipschitz of coefficient $2n$.

(c). Note that χ_E is a measurable and integrable function. By Lebesgue's differentiation theorem, for almost all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n\int_x^{1/n+x}\chi_E(t)dt=\chi_E(x)$$

as desired.

(d). By part (a), we may employ the dominated convergence theorem; by part (c), we see:

$$\begin{aligned}\lim_{n\to\infty}\int_0^1|f_n(x)-\chi_E(x)|dx &= \int_0^1\lim_{n\to\infty}|f_n(x)-\chi_E(x)|dx\\ &= 0\end{aligned}$$

So that $||f_n-\chi_E||_1 \to 0$.

5. Problem 5

We see:

$$\begin{aligned}\int_0^1 g(x)dx &= \int_0^1\int_x^1\frac{f(t)}{t}dtdt\\ &= \int_0^1\frac{f(t)}{t}\int_0^t dxdt\\ &= \int_0^1 f(t)dt\end{aligned}$$

Since $f \in L_1$, f is absolutely integrable; since $\int_0^1 g(x)dx=\int_0^1 f(x)dx$, obviously $g(x)$ must also be absolutely integrable. To see this more clearly, let $\epsilon > 0$. By absolute integrability of f, there exists δ such that $\mu(E) < \delta$ implies

$$\int_E |f(x)|dx < \epsilon$$

Then, since the integrals of f and g coincide,

$$\int_E |g(x)|dx < \epsilon$$

as desired.

6. Problem 6

(a). Since $f_n \in L_p$ for every n, take

$$M := \sup_n \{||f_n||_p\}$$

This supremum is finite, since if not, $||f_n||_p \to \infty$, contradicting the integrability of f. Thus M is finite, and by definition,

$$||f_n||_p \leqslant M$$

for all $n \in \mathbb{N}$.

(b). Note first that $||f||_p \leqslant M$ as well, M as given in part (a). Let $\epsilon > 0$; there exists $N \in \mathbb{N}$ such that

$$||f_n - f||_p \leqslant \frac{\epsilon^2}{2M}$$

for all $n > N$. We then see:

$$\begin{aligned} ||f_n^2 - f^2||_{p/2} &= ||(f_n - f)(f_n + f)||_{p/2} \\ &\leqslant ||f_n - f||_p ||f_n + f||_p \quad \text{(Hölder's)} \\ &\leqslant ||f_n - f||_p \big(||f_n||_p + ||f||_p\big) \\ &< \frac{\epsilon}{(2M)^{1/2}} \cdot (2M)^{1/2} = \epsilon \end{aligned}$$

So that $||f_n^2 - f^2||_{p/2} \to 0$.

7. Problem 7

Consider $g(z) := \frac{z}{f(z)}$; we have that $\frac{|z|}{|f(z)|} \leqslant 1$ and is hence bounded. This function is holomorphic everywhere except possibly at the origin, in which case the function still remains bounded in some deleted neighborhood. By the Riemann extension theorem, we may extend g to some $\widetilde{g}$ that is entire.

This extension will remain bounded, and we deduce by Liouville's theorem that $\widetilde{g}$ will be constant. Restricting, this implies that g is also constant; that is,

$$\frac{z}{f(z)} = c$$

Taking the modulus of the above, we also find:

$$|z| \cdot |c| \geqslant |z| \implies |c| \geqslant 1$$

In which case $f(z) = cz$ for $|c| \geqslant 1$, which was to be proved.

8. Problem 8

(a). Let $z \in \mathbb{R}$. By the condition $f(\mathbb{R}) \subset \mathbb{R}$, we know that $\overline{f(z)} = f(z)$. Comparing the power series expansions, this implies

$$\sum_n a_n z^n = \sum_n \overline{a_n} z^n$$
$$\implies a_n = \overline{a_n} \text{ for all } n$$

So that $a_n \in \mathbb{R}$ for all n.

(b). Again let $z \in \mathbb{R}$. If $f(i\mathbb{R}) \subset i\mathbb{R}$, then we deduce that $f(iz) = -\overline{f(iz)}$. Using this,

$$\sum_n a_n i^n z^n = -\sum_n a_n (-i)^n z^n$$
$$\implies a_n = (-1)^{n+1} a_n \text{ for all } n$$
$$\implies a_n = 0 \text{ for even } n$$

In which case we deduce that f is an odd function, that is, $f(-z) = -f(z)$ for all z.

9. Problem 9

(a). **False**. No such set can exists. To see this, recall that Lebesgue's differentiation theorem implies that for all $x \in E$,

$$\lim_{r \to 0} \frac{\mu(E \cap B(x,r))}{|B(x,r)|} = 1$$

However, by the condition on E, we see that the left hand side has value 1/2, so that if E had the desired property, this limit would equal 1/2, a contradiction.

(b). **True**. Recall that for f differentiable a.e, we always have that $\int_a^b f'(x)dx \leqslant f(b) - f(a)$. Then,

$$\begin{aligned} & 1 \leqslant f'(x) \\ \implies & \int_0^x dx \leqslant \int_0^x f'(t)dt \\ \implies & x \leqslant \int_0^x f'(t)dt \leqslant f(x) - f(0) \\ \implies & x \leqslant f(x) - f(0) \end{aligned}$$

Since $f(0) \geqslant 0$, we know that $f(x) - f(0) \leqslant f(x)$, so that $x \leqslant f(x)$ for all $x \in [0,1]$.

(c). **False**. Merely take

$$f_n := \chi_{[0,1/2]} - \chi_{(1/2,1]} \quad \text{for all } n$$

Then, $\int_0^1 f_n d\mu = 0$ for every n, but $f_n \not\equiv 0$ (really you can just take any odd function on a symmetric interval).

(d). **True**. Argue by contraposition. Suppose f^3 does not have an essential singularity at 0. Then, there exists N such that

$$\lim_{z\to 0} z^n f^3(z) = 0$$

for all $n > N$. Choose m such that $3m > N$; we see

$$\lim_{z\to 0} z^{3m} f^3(z) = \big(\lim_{z\to 0} z^m f(z)\big)^3 = 0$$

In which case we see that $f(z)$ cannot have an essential singularity at 0.

Alternatively, we may employ Casorati-Weierstrass to deduce this result; for every $\epsilon > 0$, we know that $f(B_\epsilon(0)\backslash 0)$ is dense in $\mathbb{C}$. Composing with $z \mapsto z^3$, which is continuous and surjective, this remains dense in $\mathbb{C}$. But the only such singularity with this property is essential, whence the result.

(e). **False**. Suppose that f is holomorphic; then, in some neighborhood of 0, we may write

$$f(z) = \sum_{n\geqslant 0} a_n z^n$$

In which case $f'(z) = \sum_{n\geqslant 1} n a_n z^{n-1}$ is still holomorphic at 0, so that there is no way that f' can have a pole of order 1.