AUGUST 2012 ANALYSIS QUALIFYING EXAM

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1. Problem 1

Let (X,ρ), (Y,σ) denote our metric spaces. Let $\epsilon > 0$; pick $K_{\epsilon/3}$ as in the problem statement. We have 3 cases:

Case 1: x, $y \in K_{\epsilon/3}$. Since f is continuous on a compact set, f is uniformly continuous on this set already.

Case 2: x, $y \notin K_{\epsilon/3}$. Then, for all M, $\rho(x,y) < M$ implies $\sigma(f(x), f(y)) < \epsilon/3 < \epsilon$.

Case 3: $x \in K_{\epsilon/3}$, $y \notin K_{\epsilon/3}$. Since f is uniformly continuous on $K_{\epsilon/3}$, we may find δ_1 such that for all a, $b \in K_{\epsilon/3}$, $\rho(a,b) < \delta_1 \implies \sigma(f(a), f(b)) < \epsilon/3$. Let $\rho(x,y) < \delta_1/2$.

Then, in particular, $\rho(y, K_{\epsilon/3}) > 0$ by compactness, and we may find $z \in \partial K_{\epsilon/3}$ such that $\rho(y,z) < \epsilon/3$. Then, note that

$$\rho(x,z) \leqslant \rho(x,y) + \rho(y,z) < \delta_1$$

Now, by continuity of f, there exists δ_2 such that for all ℓ such that $\rho(\ell, z) < \delta_2$, $\sigma(f(\ell), f(z)) < \epsilon/3$. As $z \in \partial K_{\epsilon/3}$, there exists $\ell \notin K_{\epsilon/3}$ such that $\rho(\ell, z) < \delta_2$. Then, putting this all together,

$$\sigma(f(x), f(y)) \leqslant \sigma(f(x), f(z)) + \sigma(f(z), f(\ell)) + \sigma(f(\ell), f(y))$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Whence we deduce that f is uniformly continuous.

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2. Problem 2

Suppose that $||\chi_{E_n} - f||_1 \to 0$. In particular, by Chebyshev's inequality, $\chi_{E_n} \to f$ in measure, so that we may choose a subsequence $\chi_{E_{n_k}}$ such that

$$\chi_{E_{n_k}} \to f \text{ a.e}$$

Now, for any given $x \in \mathbb{R}$, $\chi_{E_{n_k}}(x) \in \{0,1\}$, so that in order for such a subsequence to converge, it must become eventually constant for almost every x. That is, $f(x) \in \{0,1\}$ for almost every x. Now, set $E := \bigcup_{m\geqslant 0}\bigcap_{n\geqslant m}\{x \mid \chi_{E_{n_k}}(x) = 1\}$. This is certainly measurable as the union and intersection of measurable sets. By construction, if $x \in E$, then $\chi_{E_{n_k}}(x) = 1$ for all k sufficiently large, whence $\chi_E = f$ almost everywhere, as contended.

3. Problem 3

Let $N \in \mathbb{N}$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} |\sum_{n=1}^{N} f_n(x)|dx &\leqslant \sum_{n=1}^{N}\int_{\mathbb{R}} |f_n(x)|dx \\
&= \sum_{n=1}^{N}\int_{\mathbb{R}} \frac{|f(u)|}{n^{\alpha+1}}du \quad (u = nx) \\
&\leqslant ||f||_1 \cdot \sum_{n=1}^{N} \frac{1}{n^{\alpha+1}}
\end{aligned}$$

Now, let $N \to \infty$; we find

$$\int_{\mathbb{R}} \Big|\sum_{n=1}^{\infty}\Big|dx \leqslant ||f||_1 \sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}} < \infty$$

where $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}$ converges since $\alpha > 0$. This then tells us that $\sum_{n=1}^{\infty} f_n(x)$ is integrable, hence finite almost everywhere. That is, $\sum_{n=1}^{\infty}$ is convergent almost everywhere.

4. Problem 4

(a). Let $N \in \mathbb{N}$. Note that

$$f_n(x) = \int_0^x f_n'(t)dt$$

Then,

$$\begin{aligned}\Big|\sum_{n=1}^{N} f_n(x)\Big| &= \Big|\sum_{n=1}^{N}\int_0^x f_n'(t)dt\Big| \\ &\leqslant \sum_{n=1}^{N}\int_0^x |f_n'(t)|dt \\ &\leqslant \sum_{n=1}^{\infty}\int_0^1 |f_n'(t)|dt < \infty\end{aligned}$$

Letting $N \to \infty$, this series is bounded, hence convergent for all $x \in [0,1]$.

(b). Let $\epsilon > 0$. We may find $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} f_n(x) < \sum_{n=1}^{N} f_n(x) + \epsilon/4$$

Similarly, by absolute continuity of each f_n, we may find δ_n such that for all intervals $\{(a_k, b_k)\}$ with $\sum_k b_k - a_k < \delta_n$,

$$\sum_k |f_n(b_k) - f_n(a_k)| < \frac{\epsilon}{2N}$$

Then, choose $\delta := \min_n\{\delta_n\}$. Then, for any set of open intervals with $\sum_k b_k - a_k < \delta$,

$$\begin{aligned}\sum_k |f_n(b_k) - f_n(a_k)| &< \sum_k \sum_{n=1}^{N} |f_n(b_k) - f_n(a_k)| + \epsilon/2 \\ &= \sum_{n=1}^{N}\sum_k |f_n(b_k) - f_n(a_k)| + \epsilon/2 \\ &< \sum_{n=1}^{N} \frac{\epsilon}{2N} + \epsilon/2 = \epsilon\end{aligned}$$

So that f is absolutely continuous.

(c). By splitting into positive and negative parts, we may assume $f_n' \geqslant 0$. Then, using that fact that f is absolutely continuous by part (b),

$$\begin{aligned}\int_0^1 \lim_{N\to\infty} f'(1) - \sum_{n=1}^N f_n'(x)dx &\leqslant \liminf_{N\to\infty} \int_0^1 f'(x) - \sum_{n=1}^N f_n'(x)dx \\ &= \liminf_{N\to\infty} f(1) - \sum_{n=1}^N f_n'(1) \\ &= f(1) - f(1) = 0\end{aligned}$$

5. Problem 5

If $\mu(A) = \mu(B)$, then $\mu(B\backslash A) = 0$. Since $E\backslash A \subset B\backslash A$ and Lebesgue measure on $\mathbb{R}$ is a complete measure, we deduce that $E\backslash A$ is measurable and has measure 0. Then,

$$E = (E\backslash A) \cup A$$

is the union of measurable sets, hence measurable, and

$$\mu(E) = \mu(E\backslash A) + \mu(A) = \mu(A)$$

6. Problem 6

(a). Let $f \in L^2(\mathbb{R})$. Then, employing Hölder's inequality,

$$\begin{aligned}||\chi_{[-1,1]}f||_1 &\leqslant ||\chi_{[-1,1]}||_2||f\chi_{[-1,1]}||_2 \\ &= \sqrt{2}||f\chi_{[-1,1]}||_2\end{aligned}$$

b). Using part (a), we have:

$$\begin{aligned}||f||_1 &\leqslant ||f\chi_{[-1,1]}||_1 + ||f\chi_{\mathbb{R}\backslash[-1,1]}||_1 \\ &\leqslant \sqrt{2}||f||_2 + ||f\chi_{\mathbb{R}\backslash[-1,1]}||_1\end{aligned}$$

Let us now consider the rightmost term. We see:

$$\begin{aligned}
||f\chi_{\mathbb{R}\setminus[-1,1]}||_1 &= \int_{\mathbb{R}\setminus[-1,1]} f d\mu \\
&= \int_{\mathbb{R}\setminus[-1,1]} \frac{g(x)}{x} d\mu \\
&\leqslant \Big(\int_{\mathbb{R}\setminus[-1,1]} \frac{1}{x^2} dx\Big)^{1/2} ||g||_2 \quad \text{(Hölder's)} \\
&= \sqrt{2}||g||_2
\end{aligned}$$

Combining this with the above, we find:

$$||f||_1 \leqslant \sqrt{2}\big(||f||_2 + ||g||_2\big)$$

as asserted.

7. Problem 7

Since $|p_n| \to \infty$, this sequence cannot have any accumulation point. Consider now $\frac{1}{f}$. This is bounded and holomorphic away from $\{p_n\}$, so that by the Riemann extension theorem we may choose an extension $\widetilde{f}$ such that $\frac{1}{\widetilde{f}}$ extends $\frac{1}{f}$.

Then, this extension is bounded and entire, hence constant by Liouville's theorem. This then gives that $\frac{1}{f}$ is constant, so that f is constant as well, as desired.

8. Problem 8

Recall that by the Cauchy-Hadamard theorem, $\limsup_{n\to\infty} |a_n|^{1/n} = 1/R$, where R denotes the radius of convergence. We then want to solve

$$\cos(\pi z) = 0 \implies z = (2k+1)/2,\ k \in \mathbb{N}$$

Then, the closest singularities to the point $z = 1$ are located at both $-1/2$ and $1/2$. Both of these have distance

$$|i - 1/2| = \sqrt{1 + 1/4} = \frac{\sqrt{5}}{2}$$

so the radius of convergence is $\sqrt{5}/2$. Taking the reciprocal,

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{2}{\sqrt{5}}$$

9. Problem 9

(a). True. Make a change of variable such that $I + a = [-\pi, \pi]$. Then,

$$\begin{aligned}\int_I f(x)dx &= \int_{-\pi}^{\pi} f(x-a)dx \\ &= \int_{-\pi}^{\pi} f(x)dx\end{aligned}$$

Where, by periodicity, $f(x)$ and $f(x-a)$ attain the same values on the interval $[-\pi, \pi]$, whence have the same integral over $[-\pi, \pi]$.

(b). True. By maximum modulus principle,

$$\max_{|z|=1} |P(z)| \geqslant \max_{z\in B_1(0)} |P(z)|$$

Note that $P(0) = 1$, so that

$$1 \leqslant \max_{|z|=1} |P(z)|$$

as contended.

(c). False. Suppose $f(1/n) = 1/n^3$. Since $\{1/n\}_{n\in\mathbb{N}}$ has an accumulation point, the identity principle implies that $f(z) = z^3$ everywhere. However, this then means that $f(-1/n) = -1/n^3$, a contradiction.

(d). False. Set $f_n = \frac{\chi_{[0,n]}}{n}$. Then, it is easy to see that

$$||f_n||_2 = \frac{1}{n} \to 0$$

However,

$$||f_n||_1 = 1 \text{ for all } n \in \mathbb{N}$$

(e). False. Consider $f_n := \frac{\chi_{[-n,n]}}{2n}$, $f = 0$. Then,

$$|f_n - f| < \frac{1}{2n} < \frac{1}{n}$$

for all x. However,

$$||f_n||_1 = 1 > 0$$

for all $n \in \mathbb{N}$.