AUGUST 2011 ANALYSIS QUALIFYING EXAM

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1. Problem 1

Let us first show that this is not a contraction. Note that

$$\begin{aligned} ||\Omega(\chi_{[0,1]}) - \Omega(0)||_\infty &= \sup_{x\in[0,1]} \Big| \int_0^x \chi_{[0,1]} dx \Big| \\ &= \sup_{x\in[0,1]} |x| \\ &= 1 = ||\chi_{[0,1]} - 0||_\infty \end{aligned}$$

Then, by definition, Ω is not a contraction. However, note that:

$$\begin{aligned} \Omega \circ \Omega(\phi)(x) &= \int_0^x \int_0^x \phi(t) dt \\ &= \int_0^x (x-t)\phi(t) dt \end{aligned}$$

Then, let ϕ, $\psi \in C[0,1]$. We see:

$$\begin{aligned} ||\Omega^2(\phi) - \Omega^2(\psi)||_\infty &= \sup_{x\in[0,1]} \Big| \int_0^x (x-t)\big(\phi(t) - \psi(t)\big) dt \Big| \\ &\leqslant \sup_{x\in[0,1]} \Big(\int_0^x (x-t) dt \Big) ||\phi - \psi||_\infty \quad \text{(Hölder's)} \\ &= \sup_{x\in[0,1]} \frac{x^2}{2} ||\phi - \psi||_\infty \\ &= \frac{1}{2} ||\phi - \psi||_\infty \end{aligned}$$

So that Ω^2 is a contraction with $\lambda = 1/2$.

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2. Problem 2

By definition of supremum, for all $n \in \mathbb{N}$, there exist a_n, $b_n \in E$ such that

$$\rho(a_n, b_n) > \sup_{x,y} \rho(x,y) - 1/n$$

By compactness of $E \times E$, we may choose a convergent subsequence $(a_{n_k}, b_{n_k}) \to (a,b) \in E \times E$. By construction,

$$\rho(a_{n_k}, b_{n_k}) > \sup_{x,y} \rho(x,y) - \frac{1}{n_k}$$

So that as $k \to \infty$, we see

$$\rho(a,b) = \sup_{x,y} \rho(x,y)$$

as contended.

3. Problem 3

This is uniformly convergent everywhere on $[0, \infty)$. To see this, note that

$$\frac{x}{n + n^3x^3} \leqslant \frac{1}{2^{1/3} \cdot n^{2/3}} \frac{1}{n + n/2} = \frac{2^{2/3}}{3} \frac{1}{n^{5/3}}$$

In which case

$$\sum_{n=1}^{\infty} \frac{x}{n + n^3x^3} \leqslant \sum_{n=1}^{\infty} \frac{2^{2/3}}{3} \frac{1}{n^{5/3}} < \infty$$

So that by the Weierstrass M-test, this series converges uniformly everywhere.

4. Problem 4

(i). Let $A_n := \{x \mid |f(x)| > n\}$. Then,

$$\begin{aligned} \int_0^1 |f(x)| dx &= \int_{A_n} |f(x)| dx + \int_{A_n^c} |f(x)| dx \\ &\geqslant nm(A_n) + \int_{A_n^c} f(x) dx \end{aligned}$$

Note that $f\chi_{A_n^c} \to f$ since any integrable function is finite almost everywhere, so that by the monotone convergence theorem,

$$\lim_{n\to\infty} \int_{A_n^c} |f(x)|dx = \int_0^1 f(x)dx$$

Then,

$$\int_0^1 |f(x)|dx \geqslant \limsup_{n\to\infty} nm(A_n) + \limsup_{n\to\infty} \int_{A_n^c} |f(x)|dx$$
$$\implies \limsup_{n\to\infty} nm(A_n) = 0$$

In which case we deduce $\lim_{n\to\infty} nm(A_n) = 0$, as asserted.

(b). Set

$$f(x) := \begin{cases} \frac{-1}{x\log(x)}, & x \in (0, e^{-1}) \\ 0 & \text{else} \end{cases}$$

Then, $f \notin L^1(0,1)$ as f has antiderivative $\log(|\log(x)|)$. However, given $n \in \mathbb{N}$, let x_n be such that

$$\frac{-1}{x_n \log(x_n)} = n$$

Then, $nx_n = \frac{-1}{\log(x_n)}$, and, as $n \to \infty$, $x_n \to 0$, in which case we must have $-\log(x_n) \to \infty$. But then,

$$nm(A_n) = nx_n \to 0 \text{ as } n \to \infty$$

Yet $f \notin L^1(0,1)$, so we have a counterexample as desired.

5. Problem 5

Let $\epsilon > 0$. By absolute continuity of integration, there exists δ such that $\mu(S) < \delta$ implies $\int_S |f(x+h)-f(x)|dx < \epsilon/2$. By Lusin's theorem, there exists a closed set F such that $\mu(F) < \delta$ and f is continuous on F^c. Note also that

$$\int_0^1 |f(x+h) - f(x)|dx \leqslant 2||f||_1$$

so that by absolute continuity and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{h\to 0}\int_0^1 |f(x+h)-f(x)|dx &= \lim_{h\to 0}\int_F |f(x+h)-f(x)|dx \\ &+ \lim_{h\to 0}\int_{F^c} |f(x+h)-f(x)|dx \\ &< \epsilon/2 + \int_{F^c}\lim_{h\to 0}|f(x+h)-f(x)|dx \\ &= \epsilon/2 < \epsilon\end{aligned}$$

As ϵ is arbitrary, we deduce that

$$\lim_{h\to 0}\int_0^1 |f(x+h)-f(x)|dx = 0$$

6. Problem 6

Note first that $f_n\chi_E \to f\chi_E \leqslant f \in L^1$. By Lebesgue's dominated convergence theorem, we have

$$\int f_n\chi_E dx \to \int f\chi_E dx$$

that is, $\int_E f_n dx \to \int_E f dx$, as contended.

7. Problem 7

Note that by Cauchy's integral formula,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{e^z - e^{-z}}{z^4}dz &= \frac{1}{6}\cdot\frac{d^3}{dz^3}\big(e^z - e^{-z}\big)(0) \\ &= \frac{1}{6}\cdot(e^0 + e^0) = \frac{1}{3}\end{aligned}$$

8. Problem 8

Define $g_n := \inf_{k\geqslant n} f_n$, where f_n is our sequence of functions. Obviously $g_n \leqslant f_n$, so that

$$\int_E g_n \leqslant \int_E f_n$$

Since this in fact holds for all n, we have the stronger inequality:

$$\int_E g_n \leqslant \inf_{k \geqslant n} \int_E f_k$$

Note that g_n is an increasing sequence of functions. By Lebesgue's monotone convergence theorem,

$$\lim_{n\to\infty} \int_E g_n = \int_E \lim_{n\to\infty} g_n$$

Taking the limit in our inequality then yields:

$$\int_E \liminf_{n\to\infty} f_n \leqslant \liminf_{n\to\infty} \int_E f_n$$

And Fatou's Lemma is proved.

9. Problem 9

(a). True. By the Arithmetic Geometric mean inequality,

$$\limsup_{n\to\infty} (x_1 \dots x_n)^{1/n} \leqslant \limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}$$

And, by the Cesaro-Stolz theorem,

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \leqslant \limsup_{n\to\infty} x_n$$

so that

$$\limsup_{n\to\infty} (x_1 \dots x_n)^{1/n} \leqslant \limsup_{n\to\infty} x_n$$

(b). False. If such an f existed, Bessel's inequality gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \leqslant \int_{-\pi}^{\pi} |f(x)|^2 dx$$

so that $f \notin L^2(-\pi, \pi)$.

(c). True. Let f be as given. Then, using absolute continuity,

$$\begin{aligned} |f(b)-f(a)| &= \Big| \int_a^b f'(x)dx \Big| \\ &\leqslant \int_a^b |f'(x)|dx \\ &\leqslant ||f'||_p |b-a|^{1-1/p} \quad \text{(Hölder's)} \end{aligned}$$

Whence we may take $C := ||f'||_p$.

(d). False. Take $f(z) := z$. In order for this to be true, we need that f must have no zeroes on the interior.

(e). False. If $\inf\{|b-a| \mid a \in A, \ b \in B\} = 0$, then, by definition of infimum there exists $a_n \in A$ such that

$$d(a_n, B) < frm[o]--/n$$

By compactness we may choose a convergent subsequence $a_{n_k} \to a \in A$. Then, it is clear that a satisfies $d(a, B) = 0$, in which case $a \in \overline{B}$. Since B is closed, $B = \overline{B}$, so that $a \in B$, which is a contradiction to disjointness.