AUGUST 2009 ANALYSIS QUALIFYING EXAM

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1. Problem 1

Confer problem 1 of the August 2012 qualifying exam for a more general version of this problem.

We may find $N \in \mathbb{N}$ such that for all $|x| > N$, $f(x) < \epsilon/2$. Then, $f|_{[-N,N]}$ is uniformly continuous by compactness of $[-N,N]$. Similarly, $f|_{\mathbb{R}\setminus[-N,N]}$ is uniformly continuous since for all x, $y \in \mathbb{R}\setminus[-N,N]$,

$$|f(x) - f(y)| \leqslant |f(x)| + |f(y)|$$
$$< \epsilon$$

It remains only to show uniformity in the "boundary" case. We ay choose $M \in \mathbb{N}$ such that for all $|y| > M$, $|f(y)| < \epsilon/6$.

By uniform continuity of $f|_{[-M,M]}$, there exists δ_1 such that for all a, $b \in [-M,M]$,

$$|a-b| < \delta_1 \implies |f(a) - f(b)| < \epsilon/3$$

Suppose now that $x \in [-M,M]$, $y \notin [-M,M]$ with $|x-y| < \delta_1/2$. Then, we may also find $z \in \partial[-M,M]$ with $|y-z| < \delta_1/2$, whence

$$|x-z| \leqslant |x-y| + |y-z| < \delta_1$$

By continuity of f, there exists δ_2 such that for all ℓ with $|\ell - z| < \delta_2$, $|f(\ell) - f(z)| < \epsilon/3$. As $z \in \partial[-M,M]$. we may find $\ell \notin [-M,M]$ with

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$|\ell - z| < \delta_2$, so that

$$\begin{aligned}|f(x) - f(y)| &\leqslant |f(x) - f(z)| + |f(z) - f(\ell)| + |f(\ell) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon\end{aligned}$$

In which case f is uniformly continuous.

2. Problem 2

Since $f^n \in L^1(X)$ for all n, we deduce that $\lim_{n\to\infty} f^n$ is finite almost everywhere; that is $f^n \leqslant 1$ almost everywhere. Now, set

$$S := \{x \mid |f(x)| < 1\}$$

If $m(S) > 0$, then

$$\begin{aligned}\int_X f dx &= \int_X f^n dx \\ &= \int_S f^n dx + \int_{S^c} dx \\ &< \int_{S^c} f + \int_S f dx \\ \implies \int_X f^n dx &< \int_X f dx\end{aligned}$$

Which is a contradiction to our assumption, in which case we must have $m(S) = 0$; then, combining this with the above observation, we have that $f \equiv 1$ almost everywhere. Set

$$E := \{x \mid f(x) = 1\}$$

The above is clearly measurable by writing $E = (\{x \mid f(x) < 1\} \cup \{x \mid f(x) > 1\})^c$. By construction, we also have that

$$f|_E \equiv 1 \equiv \chi_E|_E$$

and, as E^c is the union of two null sets, E^c has measure 0, whence $f = \chi_E$ almost everywhere, as contended.

3. Problem 3

By absolute continuity, we may write

$$|f(x) - f(y)| = \Big| \int_y^x f'(t)dt \Big|$$

Then,

$$\begin{aligned} |f(x) - f(y)| &= \Big| \int_y^x f'(t)dt \Big| \\ &\leqslant \int_y^x |f'(t)|dt \\ &\leqslant ||f'||_2 \cdot |x-y|^{1/2} \quad \text{(Hölder's)} \end{aligned}$$

Whence we may take $M := ||f'||_2$.

4. Problem 4

(a). Note that f is continuous and $[0,1]$ is compact by Heine-Borel. Any continuous f on a compact set is uniformly continuous, so that f is uniformly continuous on $[0,1]$.

(b). Observe the the inequality

$$|\sqrt{x} - \sqrt{y}|^{1/2} \leqslant |\sqrt{x} + \sqrt{y}|^{1/2}$$

is trivially true by the triangle inequality. Now, simply multiply the above by $|\sqrt{x} - \sqrt{y}|^{1/2}$:

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &\leqslant |\sqrt{x} + \sqrt{y}|^{1/2} \cdot |\sqrt{x} - \sqrt{y}|^{1/2} \\ &= |x-y|^{1/2} \end{aligned}$$

In which case

$$|f(x) - f(y)| \leqslant |x-y|^{1/2}$$

(c). $f'(x) = \frac{1}{2\sqrt{x}}$, and,

$$||f'||_2 = \frac{1}{4} \int_0^1 \frac{1}{x} dx = \infty$$

so that $f' \notin L^2(0,1)$.

5. Problem 5

Assume first (by splitting into positive and negative parts) that f is nonnegative. Then, note that

$$(\mu \times \mu)(E) = \int_E d(\mu \times \mu)$$

Now, set

$$S := \{(x,t) \mid 0 \leqslant t < f(x)\}$$

$$S' := \{(x,t) \mid 0 \leqslant t \leqslant f(x)\}$$

Obviously $E = S' \backslash S$ by definition, and,

$$\begin{aligned}\int_S d(\mu \times \mu)(x,t) &= \int_{\mathbb{R}} \int_0^{f(x)} d\mu(t) d\mu(x) \\ &= \int_{\mathbb{R}} f(x) d\mu(x)\end{aligned}$$

$$\begin{aligned}\int_{S'} d(\mu \times \mu)(x,t) &= \int_{\mathbb{R}} \int_0^{f(x)} d\mu(t) d\mu(x) \\ &= \int_{\mathbb{R}} f(x) d\mu(x)\end{aligned}$$

In which case

$$\begin{aligned}\int_E d(\mu \times \mu)(x,t) &= \int_{S' \backslash S} d(\mu \times \mu) \\ &= \int_{\mathbb{R}} f(x) d\mu(x) - \int_{\mathbb{R}} f(x) d\mu(x) \\ &= 0\end{aligned}$$

So that $(\mu \times \mu)(E) = 0$, as desired.

6. Problem 6

Make the change of variable $z = e^{ix}$, so that as x goes from 0 to 2π, we end up integrating z over the unit circle. Then,

$$dz = ie^{ix} dx \implies dx = \frac{dz}{iz}$$

And,

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \implies \sin(x) = \frac{(z^2-1)}{2iz}$$

Letting C denote the unit circle, our integral becomes

$$\int_C \frac{1}{2 - \frac{(z^2-1)^2}{4z^2}} \cdot \frac{dz}{iz} = \int_C \frac{-4izdz}{8z^2 - z^4 + 2z^2 - 1}$$
$$= \int_C \frac{4izdz}{z^4 - 10z^2 + 1}$$

Now, let us find the roots of the denominator:

$$z^4 - 10z^2 + 1 = 0 \implies z^2 = 5 \pm 2\sqrt{6}$$

In which case we have 4 roots:

$$z = \sqrt{5 + 2\sqrt{6}},\ -\sqrt{5+2\sqrt{6}},\ \sqrt{5-2\sqrt{6}},\ -\sqrt{5-2\sqrt{6}}$$

Note that the first two roots above do not lie in the unit circle, so they may be ignored when computing the integral. It remains to compute the residues at

$$\lim_{z\to\sqrt{5-2\sqrt{6}}} \frac{4i(z-\sqrt{5-2\sqrt{6}})z}{z^4-10z^2+1} = \lim_{z\to\sqrt{5-2\sqrt{6}}} \frac{4iz}{(z+\sqrt{5-2\sqrt{6}})(z^2-5-2\sqrt{6})}$$
$$= \frac{-i}{2\sqrt{6}}$$

$$\lim_{z\to-\sqrt{5-2\sqrt{6}}} \frac{4i(z+\sqrt{5-2\sqrt{6}})z}{z^4-10z^2+1} = \lim_{z\to-\sqrt{5-2\sqrt{6}}} \frac{4iz}{(z-\sqrt{5-2\sqrt{6}})(z^2-5-2\sqrt{6})}$$
$$= \frac{-i}{2\sqrt{6}}$$

Whence, by Cauchy's residue theorem,

$$\int_C \frac{4izsz}{z^4-10z^2+1} = \frac{2\pi}{\sqrt{6}}$$

Now, since the integral in the problem has bounds 0 to $\pi/2$, we simply divide the above by 4 to find

$$|int_0^{\pi/2} \frac{dx}{2+\sin^2(x)} = \frac{\pi}{2\sqrt{6}}$$

The above can actually be solved by the standard tangent half angle substitution, but I assumed that problem was intended to be done via contour integration.

7. Problem 7

Since $|p_n| \to \infty$, this sequence cannot have any accumulation point. Consider now $\frac{1}{f}$. This is bounded and holomorphic away from $\{p_n\}$, so that by the Riemann extension theorem we may choose an extension $\widetilde{f}$ such that $\frac{1}{\widetilde{f}}$ extends $\frac{1}{f}$.

Then, this extension is bounded and entire, hence constant by Liouville's theorem. This then gives that $\frac{1}{f}$ is constant, so that f is constant as well, as desired.

8. Problem 8

Since we are on a bounded domain and F is continuous, we may employ Morera's theorem (note F is continuous since f is). Let $\gamma : [0,1] \to \mathbb{C}$ be any smooth closed contour contained in Ω; we see:

$$\begin{aligned}
\int_\gamma F(z)dz &= \int_0^1 F(\gamma(s))\gamma'(s)ds \\
&= \int_0^1 \int_a^b f(\gamma(s),t)\gamma'(s)dtds \\
&= \int_a^b \int_0^1 f(\gamma(s),t)\gamma'(s)dsdt \\
&= \int_a^b \int_\gamma f(z,t)dzdt \\
&= 0
\end{aligned}$$

Where changing the order of integration is justified since both itnegrals exist and are finite, and the final equality follows by Cauchy's integral theorem. As γ was arbitrary, Morera's theorem allows us to deduce that F is holomorphic in Ω.

Now, let $\gamma : [0,1] \to \mathbb{C}$ be any smooth closed contour contained in Ω. We compute using Cauchy's integral formula:

$$\begin{aligned}F^{(n)}(z) &= \frac{n!}{2\pi i}\int_\gamma \frac{F(\xi)}{\xi - z)^{n+1}}d\xi \\ &= \frac{n!}{2\pi i}\int_0^1 \frac{F(\gamma(s)\gamma'(s)}{(\gamma(s) - z)^{n+1}}ds \\ &= \frac{n!}{2\pi i}\int_0^1\int_a^b \frac{f(\gamma(s),t)\gamma'(s)}{(\gamma(s) - z)^{n+1}}dtds \\ &= \frac{n!}{2\pi i}\int_a^b\int_0^1 \frac{f(\gamma(s),t)\gamma'(s)}{(\gamma(s) - z)^{n+1}}dsdt\end{aligned}$$

$$\int_a^b \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi,t)}{(\xi - z)^{n+1}}d\xi$$

$$= \int_a^b \frac{\partial^n f}{\partial z^n}(z,t)dt$$

Which gives our derivative formula.

9. Problem 9

(a). False. Take

$$O := \bigcup_{q\in\mathbb{Q}} B_{\epsilon/2^{n+1}}(q)$$

Where $\epsilon < 1$. Then, this has measure $\leqslant \epsilon$ and is open as the union of open sets, so that

$$m([0,1]\backslash O) \geqslant 1 - \epsilon$$

And, as O contains a dense subset of $[0,1]$, it is itself dense in $[0,1]$.

(b). False. Recall that compactness in a general metric space is equivalent to being complete and totally bounded, so let us choose a non-complete space. Define $f(x) := \chi_{[0,\sqrt{2}]}(x)$ on the set $[0,2]\cap\mathbb{Q}$ with the

induced topology. Then,

$$f^{-1}(\{1\}) = [0, \sqrt{2}] \cap \mathbb{Q} \text{ is closed}$$
$$f^{-1}(\{0\}) = [\sqrt{2}, 2] \cap \mathbb{Q} \text{ is closed}$$
$$f^{-1}(\{0,\ 1\}) = [0, 2] \cap \mathbb{Q} \text{ is closed}$$

So that the preimage of every closed set of $\text{Im}(f)$ is closed, in which case f is continuous by definition[1].

Now, by density of $\mathbb{Q}$ i $\mathbb{R}$, we may find $x_n \in [0, \sqrt{2}] \cap \mathbb{Q}$ and $y_n \in [\sqrt{2}, 2] \cap \mathbb{Q}$ with

$$|x_n - \sqrt{2}| < \frac{1}{2n}, \quad |y_n - \sqrt{2}| < \frac{1}{2n}$$

Then,

$$\begin{aligned} |x_n - y_n| &\leqslant |x_n - \sqrt{2}| + |y_n - \sqrt{2}| \\ &< \frac{1}{n} \end{aligned}$$

So that $|x_n - y_n| \to 0$ as $n \to \infty$, but $|f(x_n) - f(y_n)| = 1$ for all n, so this is certainly not uniformly continuous.

(c). False. Consider the following sequence of intervals by setting $I_{j,k} = [(k-1)/j, k/j)$ for $1 \leqslant k < j$ and $I_{j,j} = [(j-1)/j, 1]$. Then, as j, $k \to \infty$, obviously $\mu(I_{j,k}) \to 0$, in which case $\mu(I_{j,k}^c) \to 1$. However, note that

$$\limsup_{j,k \to \infty} I_{j,k} = [0, 1]$$

In which case we see that every point misses infinitely often, so that certainly no subsequence will satisfy the requirement of the problem.

(d). False. The function $1/z^2$ satisfies these requirements but obviously has a pole of order 2 at 0.

[1]Note the difference between $\mathbb{R}$ and $\mathbb{Q}$, where characteristic functions are never continuous. The above proof does *not* work over $[0, 2]$, since $f^{-1}(\{0\}) = (\sqrt{2}, 2]$ is no longer closed.

(e). True.

$$||\sum_{n=1}^{\infty} f_n||_2 \leqslant \sum_{n=1}^{\infty} ||f_n||_2 < \infty$$

whence $\sum_{n=1}^{\infty} f_n$ is finite almost everywhere, so that $f_n(x) \to 0$ almost everywhere.