AUGUST 2007 ANALYSIS QUALIFYING EXAM

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1. Problem 1

Let $x_0 \in X$ and define x_n inductively by $x_n = \Omega(x_{n-1})$. Then, we can show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy as for $m > n$,

$$\begin{aligned}
\rho(x_n, x_m) &\leqslant \rho(x_m, x_{m-1}) + \cdots + \rho(x_{n+1}, x_n) \\
&\leqslant \big(\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n\big)\rho(x_1, x_0) \\
&= \frac{\lambda^n - \lambda^m}{1-\lambda}\rho(x_1, x_0) \to 0 \text{ as } m,\ n \to \infty
\end{aligned}$$

By completeness of X, we deduce that $x_n \to x \in X$. Now, consider $\Omega(x)$; we want to show that x must be a fixed point:

$$\begin{aligned}
\rho(\Omega(x), x) &\leqslant \rho(x, x_{n+1}) + \rho(x_{n+1}, \Omega(x)) \\
&\leqslant \rho(x, x_{n+1}) + \lambda\rho(x_n, x)
\end{aligned}$$

Letting $n \to \infty$ on the right, this must tend to 0, in which case

$$\rho(x, \Omega(x)) = 0$$

That is, $\Omega(x) = x$. Lastly, it remains to show uniqueness. Suppose then that x and y are two fixed points of Ω; then:

$$\rho(x, y) = \rho(\Omega(x), \Omega(y)) \leqslant \lambda\rho(x, y)$$

Since $\lambda < 1$, we must have $\rho(x, y) = 0$, so that $x = y$.

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2. Problem 2

Let $\epsilon > 0$ and set $\limsup_{n\to\infty} x_n := L$. Then, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$x_n < L + \epsilon$$

Then,

$$\begin{aligned}\sum_{k=1}^n x_k &= \sum_{k=1}^N x_k + \sum_{k=N+1}^n x_k \\ &< \sum_{k=1}^N x_k + (L+\epsilon)(n-N) \\ \implies \frac{\sum_{k=1}^n x_k}{n} &< \frac{\sum_{k=1}^N x_k}{n} + (L+\epsilon)\Big(1 - \frac{N}{n}\Big)\end{aligned}$$

Taking the limit superior of the above,

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \leqslant L + \epsilon$$

And as ϵ is arbitrary,

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \leqslant \limsup_{n\to\infty} x_n$$

As asserted.

3. Problem 3

Note that for $n \geqslant 2$,

$$\frac{x^n}{\log(n+1)} \leqslant x^n$$

so that

$$(1-x)\sum_{n=1}^\infty \frac{x^n}{\log(n+1)} \leqslant (1-x)\Big(\frac{x}{\log(2)} + \frac{x^2}{1-x}\Big)$$

Letting $x \to 1$, the above remains bounded so that our sum is bounded by a uniformly convergent series, hence itself uniformly convergent.

Now, it remains to compute M_n. We see that $x^n(1-x)$ attains its maximum at $x = \frac{n}{n+1}$, whence

$$M_n = \frac{\left(\frac{n}{n+1}\right)^n}{(n+1)\log(n+1)}$$

Note that $\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ as $n \to \infty$, and $\left(\frac{n}{n+1}\right)^n > \frac{1}{4}$ for all n. Then,

$$\sum_{n=1}^{\infty} M_n \geqslant \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$$

and, by the integral test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$ diverges so that $\sum_{n=1}^{\infty} M_n$ also diverges.

4. Problem 4

This is uniformly convergent everywhere on $[0, \infty)$. To see this, note that

$$\frac{x}{n+n^3x^3} \leqslant \frac{1}{2^{1/3} \cdot n^{2/3}} \frac{1}{n+n/2} = \frac{2^{2/3}}{3} \frac{1}{n^{5/3}}$$

In which case

$$\sum_{n=1}^{\infty} \frac{x}{n+n^3x^3} \leqslant \sum_{n=1}^{\infty} \frac{2^{2/3}}{3} \frac{1}{n^{5/3}} < \infty$$

So that by the Weierstrass M-test, this series converges uniformly everywhere.

5. Problem 5

Assume $|f(z)| \leqslant M$. By holomorphicity, we have a power series expansion

$$f(z) = \sum_{n \geqslant 0} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{B_r(0)} \frac{f(z)}{z^{n+1}} dz$$

Consider now for $n \geqslant 1$,

$$\begin{aligned} |a_n| &\leqslant \frac{1}{2\pi} \int_{B_r(0)} \frac{|f(z)|}{|z|^{n+1}} dz \\ &= \frac{1}{2\pi r^{n+1}} \int_{B_r(0)} |f(z)| dz \\ &\leqslant \frac{1}{2\pi r^{n+1}} \cdot M \cdot 2\pi r \\ &= \frac{M}{r^n} \end{aligned}$$

As f is entire, we may take $r \to \infty$ to find that $|a_n| = 0$ for all $n \geqslant 1$; that is, $f \equiv a_0$, so that f is constant.

6. Problem 6

Define $g_n := \inf_{k \geqslant n} f_n$, where f_n is our sequence of functions. Obviously $g_n \leqslant f_n$, so that

$$\int_E g_n \leqslant \int_E f_n$$

Since this in fact holds for all n, we have the stronger inequality:

$$\int_E g_n \leqslant \inf_{k \geqslant n} \int_E f_k$$

Note that g_n is an increasing sequence of functions. By Lebesgue's monotone convergence theorem,

$$\lim_{n\to\infty} \int_E g_n = \int_E \lim_{n\to\infty} g_n$$

Taking the limit in our inequality then yields:

$$\int_E \liminf_{n\to\infty} f_n \leqslant \liminf_{n\to\infty} \int_E f_n$$

And Fatou's Lemma is proved.

7. Problem 7

Suppose

$$\lim_{k\to\infty} \int_{E_k} f d\mu = \int_0^1 f d\mu$$

Then, $f\chi_{E_k} \leqslant f > 0$, so by Lebesgue's dominated convergence theorem:

$$|int_0^1 f\big(\lim_{k\to\infty}\chi_{E_k} - 1\big)edx = 0$$

As $f > 0$, Hölder's inequality gives that

$$\int_0^1 \big(\lim_{k\to\infty}\chi_{E_k} - 1\big)dx = 0$$

So that $\lim_{k\to\infty} m(E_k) = 1$.

8. Problem 8

Note that $f_n\chi_E \leqslant \sup_n f_n\chi_E \leqslant \sup_n f_n \in L^1(\mathbb{R})$, where $\sup_n f_n \in L^1(\mathbb{R})$ by assumption. By Lebesgue's dominated convergence theorem,

$$\int_E f_n d\mu = \int f_n\chi_E d\mu \to \int f\chi_E d\mu = \int_E f d\mu$$

as desired.

9. Problem 9

Replacing f and g by $f/||f||_p$ and $g/||g||_q$ respectively, we may assume by homogeneity that $||f||_p = ||g||_q = 1$ (note that if either norm vanishes the result is trivial).

By Young's inequality,

$$\begin{aligned} ||fg||_1 &= \int_E |fg|d\mu \\ &\leqslant \int_E \frac{|f|^p}{p} + \frac{|g|^q}{q}d\mu \\ &= \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

10. Problem 10

Note that

$$\int_0^1 |f(x+h) - f(x)|^p dx \leqslant 2^p ||f||_p^p < \infty$$

So by the dominated convergence theorem, we may interchange the order of the limit and integration.

Let $\epsilon > 0$. By absolute continuity of integration, there exists δ such that for all $\mu(A) < \delta$,

$$\int_A |f(x+h) - f(x)|^p dx < \epsilon$$

By Lusin's theorem, we can find a closed set F with $\mu(F) < \delta$ such that f is continuous on F^c. Then,

$$\begin{aligned}
\lim_{h\to 0} \int_0^1 |f(x+h) - f(x)|^p dx &= \lim_{h\to 0} \Big(\int_F |f(x+h) - f(x)|^p dx \\
&\quad + \int_{F^c} |f(x+h) - f(x)|^p dx \Big) \\
&< \epsilon + \lim_{h\to 0} \int_{F^c} |f(x+h) - f(x)|^p dx \\
&= \epsilon + \int_0^1 |\lim_{h\to 0} f(x+h) - f(x)|^p dx \\
&= \epsilon
\end{aligned}$$

As ϵ is arbitrary, the result follows.