AUGUST 2010 ANALYSIS QUALIFYING EXAM

KELLER VANDEBOGERT

1. Problem 1

(a). Note that $x \mapsto d(x, E)$ is a continuous function so that $O_n = f^{-1}(0, 1/n)$ is open by definition of continuity.

(b). Suppose that $x \in O_n$ for all $n \in \mathbb{N}$. Then, $d(x, E) < 1/n$ for all n, so that as $n \to \infty$, $d(x, E) = 0$ so that $x \in \overline{E}$. As E is closed, $x \in E$.

This then implies that $E = \bigcap_{n=1}^\infty O_n$. As E is compact, E is bounded by Heine-Borel, so that $m(O_n) < \infty$ for n sufficiently large. Whence

$$m(\bigcap_{n=1}^\infty O_n) = \lim_{n\to\infty} m(O_n)$$

As desired.

(c). Let $E = \mathbb{R}$ viewed as a subset of $\mathbb{R}^2$. This is closed and unbounded, yet has measure 0. However, $m(O_n) = \infty$ for all n. Similarly, enumerate $\mathbb{Q} \cap [0, 1]$ as $\{q_1, q_2, \dots\}$ and set

$$E := \bigcup_{n=1}^\infty B_{\epsilon/2^{n+1}}(q_n)$$

Obviously $\mu(E) < \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon$, but by density of $\mathbb{Q}$, $m(O_n) \geqslant 1$ for all n.

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2. Problem 2

(a). Recall that for A closed and K compact, $A+K$ is closed. Now, B can be written as the countable union of compact sets

$$B = \bigcup_{n=1}^{\infty} K_n, \ K_n \text{ compact}$$

then,

$$A+B = \bigcup_{n=1}^{\infty} (A+K_n)$$

is the countable union of closed (hence measurable) sets, so that $A+B$ is measurable.

(b). Set $A = \mathbb{Z}$, $B = \sqrt{2}\mathbb{Z}$. Then, $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ is dense in $\mathbb{R}$, so that it is certainly not closed.

3. Problem 3

(a). Note that for $u = nx$,

$$\begin{aligned}\int_{\mathbb{R}} f_n(x)dx &= \int_{\mathbb{R}} \frac{f(nx)}{n} dx \\ &= \frac{1}{n^2} \int_{\mathbb{R}} f(u)du \\ &= \frac{1}{n^2} ||f||_1\end{aligned}$$

So that each $f_n \in L^1(\mathbb{R})$.

(b). For $N \in \mathbb{N}$, consider:

$$\begin{aligned}\Big| \int_{\mathbb{R}} \sum_{n=1}^{N} f_n(x)dx \Big| &\leqslant \sum_{n=1}^{N} \int_{\mathbb{R}} |f_n(x)|dx \\ &= ||f||_1 \cdot \sum_{n=1}^{N} \frac{1}{n^2}\end{aligned}$$

Letting $N \to \infty$, we see

$$\sum_{n=1}^{\infty} ||f_n||_1 \leqslant ||f||_1 \cdot \frac{\pi^2}{6}$$

Since this sum in convergent, we deduce $||f_n||_1 \to 0$, that is, $f_n \to 0$ a.e.

4. Problem 4

" $\implies$ " Suppose first that $F(x) = \int_{-\infty}^{x} f(t)dt$. Obviously F is absolutely continuous by absolute continuity of integration. Similarly, let us compute:

$$
\begin{aligned}
TV_{-M}^{M}(F) = & \sup_{P \text{ partition}} \sum_{P} |F(x_k) - F(x_{k-1})| \\
& \sup_{P \text{ partition}} \sum_{P} \Big| \int_{x_{k-1}}^{x_k} f(x)dx \Big| \\
& \leqslant \sup_{P} \int_{-M}^{M} |f(x)|dx \\
& \leqslant ||f||_1 < \infty
\end{aligned}
$$

Finally, it is clear that $\lim_{x\to-\infty} F(x) = 0$, since $\lim_{x\to-\infty} \int_{-\infty}^{x} |f(t)|dt = 0$.

" $\impliedby$ " By absolute continuity, we have that $F(x) - F(y) = \int_{y}^{x} F'(t)dt$. Letting $y \to -\infty$, we know that $F(y) \to 0$, whence

$$F(x) = \int_{-\infty}^{x} F'(t)dt$$

Recall that the total variation can be easily computed as

$$TV_{-M}^{M}(F) = \int_{-M}^{M} |F'(t)|dt$$

Taking the supremum over all M, we see that $\int_{-\infty}^{\infty} |F'(t)|dt < \infty$ (by assumption), whence $F' \in L^1(\mathbb{R})$, which completes the proof.

5. Problem 5

This is simply a computation using Fubini-Tonelli. Set $A_t := \{x \mid |f(x)| \geqslant t\}$:

$$\begin{aligned}
\int_E |f|^p dx &= \int_E \int_0^{|f|} pt^{p-1} dt dx \\
&= \int_E \int_0^{\infty} pt^{p-1} \chi_{A_t} dt dx \\
&= p \int_0^{\infty} t^{p-1} \int_E \chi_{A_t} dx dt \quad \text{(Fubini-Tonelli)} \\
&= p \int_0^{\infty} t^{p-1} m\big(\{x \in E \mid |f(x)| \geqslant t\}\big) dt
\end{aligned}$$

As desired.

6. Problem 6

We start where the previous problem left off. We have:

$$\begin{aligned}
\int_E |f|^p dx &= p \int_0^{\infty} t^{p-1} m\big(\{x \in E \mid |f(x)| \geqslant t\}\big) dt \\
&\leqslant p \int_0^{\infty} t^{p-2} \int_E |g| \cdot \chi_{\{x \mid |f(x)| \geqslant t\}} dx dt \\
&= p \int_E |g| \int_0^{|f|} t^{p-2} dt dx \quad \text{(Fubini-Tonelli)} \\
&= \frac{p}{p-1} \int_E |g| \cdot |f|^{p-1} dx \\
&\leqslant \frac{p}{p-1} \Big(\int_E |g|^p dx \Big)^{1/p} \Big(\int_E |f|^p dx \Big)^{1-1/p} \quad \text{(Hölder's)} \\
\implies \Big(\int_E |f|^p dx \Big)^{1/p} &\leqslant \frac{p}{p-1} \Big(\int_E |g|^p dx \Big)^{1/p}
\end{aligned}$$

As contended.

7. Problem 7

Note first that since $\frac{\cos(x)}{(1+x^2)^2}$ is an even function,

$$\int_{0\infty} \frac{\cos(x)}{(1+x^2)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)^2}dx$$

Then, we shall compute the latter integral by finding the real part of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)^2}dx$$

Consider the standard upper half semicircle contour of radius R, which will be denoted by C. We have a single pole of order 2 at $z = i$ contained within this contour. Then, we can compute the residue at this point:

$$\begin{aligned}\mathrm{Res}\Big(\frac{e^{iz}}{(1+z^2)^2}, i\Big) &= \lim_{z\to i} \frac{d}{dz}\frac{e^{iz}}{(z+i)^2} \\ &= \lim_{z\to i} \frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3} \\ &= -i\Big(\frac{1}{2e}\Big)\end{aligned}$$

So that by the Residue theorem,

$$\int_C \frac{e^{iz}}{(1+z^2)^2}dz = \frac{\pi}{e}$$

Now, split the contour into two parts; C_1 will denote the section of the contour on the real line and C_2 will denote the arc in the upper half plane. Now, on the upper half plane, $\left|e^{iz}\right| \leqslant 1$, so that as $R \to \infty$,

$$\Big|\int_{C_2} \frac{e^{iz}}{(1+z^2)^2}dz\Big| \leqslant \frac{\pi R}{(1+R^2)^2} \to 0$$

And we are left with

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)^2}dz = \frac{\pi}{e}$$

And, taking real parts and dividing by 2, we find

$$\int_{0\infty} \frac{\cos(x)}{(1+x^2)^2}dx = \frac{\pi}{2e}$$

8. Problem 8

Assume f has no zeroes. Then, by maximum modulus principle,

$$|f(z)| \leqslant \max_{z \in \partial G} |f(z)| = c$$

If $c = 0$, then $f \equiv 0$ and the result is trivial, so assume $c > 0$. If f has no zeroes, then $1/f$ is also holomorphic in G so that another application of maximum modulus gives

$$\left| \frac{1}{f(z)} \right| \leqslant \max_{z \in \partial G} \left| \frac{1}{f(z)} \right| = \frac{1}{c}$$

Whence $|f(z)| \geqslant c$. But then $c \leqslant |f(z)| \leqslant c$ everywhere in G, in which case we deduce that f is identically constant. Now, if f does have zeroes, then f is not necessarily constant; consider $f(z) := z$ on the unit disk.

9. Problem 9

(a). False. Let $f(x) = e^x$. Then,

$$f(x + 1/n) - f(x) = e^x(e^{1/n} - 1)$$

Since $e^{1/n} - 1 > 0$, we may choose x very large to make the above difference arbitrarily large.

(b). False. By Cauchy-Hadamard, this implies that the radius of convergence is $1/2$. However, if f is holomorphic in the unit disk, the radius of convergence is obviously at least 1.

(c). False. Choose any nonmeasurable subset E of $[0, 1]$ and set

$$f(x) := \begin{cases} 1, & x \in E \\ -1, & x \notin E \end{cases}$$

Obviously f is not measurable, however, $|f|$ is simply the constant function 1, which is clearly measurable.

(d). True. Every measurable set may be approximated from below by closed sets. By σ-finiteness, we may further arrange that these sets are bounded, hence compact by Heine-Borel.

(e). **True**. Recall that for F differentiable a.e, we always have that $\int_a^b F'(x)dx \leqslant F(b) - F(a)$. Then,

$$\begin{aligned} &1 \leqslant F'(x) \\ \implies &\int_0^x dx \leqslant \int_0^x F'(t)dt \\ \implies &x \leqslant \int_0^x F'(t)dt \leqslant F(x) - F(0) \\ \implies &x \leqslant F(x) - F(0) \end{aligned}$$

Since $F(0) = 0$, $x \leqslant F(x)$ for all $x \in [0, 1]$.