PI AND FOURIER SERIES

KELLER VANDEBOGERT AND CHARLEY JOYNER

Abstract. In many infinite series, π seems to have a mysterious relation with the solutions of these infinite series. In the past, many ad hoc methods such as Euler's solution to the Basel Problem were employed, but the elegant theory of Fourier Series gives a more rigorous foundation to these solutions and helps explain how π seems to always pop up. We will introduce the trigonometric Fourier Series and use it and Parseval's identity to solve the Basel Problem. Then, we will present as motivation the original method of solution for the 1-dimensional Heat Equation and how the study of Fourier series naturally arises in the solution of partial differential equations, spawning a discussion of separable Hilbert Spaces. An abstract form of the generalized Fourier series by means of eigenvector expansion is then stated and proved, from which the trigonometric Fourier series is deduced as a simple corollary.

1. Introduction

Convergence of infinite series has always been an interesting area of study. There are very standard, rather low level methods for showing that a series converges. Often, convergence is all that is asked for, and the actual limiting value is not considered important. For example, consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

It is a very simple task to show that this series does in fact converge. However, *what does it converge to*? This question is significantly harder, and the solution of this particular series is known as

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the *Basel Problem*. Indeed, some of the greatest minds of mathematics have devoted significant effort to solving these problems. Newton, Euler, Machin, and even Riemann are some of the most notable examples. Their methods ranged from extremely clever geometric arguments, Taylor series expansions, all the way to very sophisticated techniques from Complex Analysis. In fact, the aptly named *Riemann Zeta Function* has spawned a problem so deep that it is currently one of the million dollar problems on the Clay Institute's list. It is amazing that such a natural question has such far reaching and complex connections to all of mathematics.

1.1. **Expansions of π.** Indeed, π seems to have a deep connection with a multitude of infinite series. Here are just a few examples:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + ...$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1*2}{3*5} + \frac{1*2*3}{3*5*7} + ...$$

and, finally:

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ...$$

While these expansions are certainly interesting and elegant to look at, how would one go about deriving them? This question will lead us into the world of Fourier series as one possible method of explanation.

2. The Trigonometric Fourier Series

The trigonometric Fourier series is often studied in an undergraduate course on ordinary differential equations. We will present it here without proof, so it can be used to solve the Basel Problem. Also, let $L^2(a, b)$ denote the set of square integrable functions over the interval (a, b).

Theorem 2.1. *Let $f \in L^2(a, b)$. Then f can be written as an infinite series of the following form:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

where

$$a_0 = \frac{2}{b-a}\int_a^b f(x)dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2n\pi x}{b-a}\right)dx$$

This representation is called the Fourier Series for f.

With this representation, we can derive the solution of the Basel Problem, although we need one more small lemma.

Lemma 2.2 (Parseval's Equality)**.** *If $f \in L^2(a, b)$ and*

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

where the coefficients are defined as in Theorem 2.1, then

$$\frac{2}{b-a}\int_a^b f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Both of the above formulas will be proved in a more abstract setting later on. For now, we will solve the Basel Problem.

Theorem 2.3.

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ...$$

Proof. Consider $f(x) = x$ on the symmetric interval $(-\pi, \pi)$. It is clear that f is bounded on this interval, and thus $f \in L^2(-\pi, \pi)$. We now consider the Fourier series of f given by Theorem 2.1. We start by computing our coefficients. This is where we can use the properties of f to our advantage. Since f is an odd function on a symmetric interval, we see that a_0 vanishes. Since an odd function multiplied by an even is *also* an odd function, we see that our coefficients a_n will also vanish. Thus, we only need to consider b_n.

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} x \sin\Big(\frac{2n\pi x}{2\pi}\Big) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx \end{aligned}$$

This integral can be solved by parts. We have:

$$\begin{aligned} b_n &= \frac{2}{\pi}\Big(\big[\frac{-x}{n} \cos(nx)\big]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx) dx \Big) \\ &= \frac{2}{\pi}\Big(\frac{-\pi}{n} \cos(n\pi)\Big) \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

Where we note on the last step that $\cos(n\pi) = (-1)^n$. Thus, we have that:

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \tag{2.1}$$

We now employ Parseval's inequality to get to a more familiar form. By Lemma 2.2:

$$\frac{2}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \Big(\frac{2(-1)^{n+1}}{n}\Big)^2$$
$$= \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

We now divide both sides by 4, and find:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{2.2}$$

As desired.

□

Remark 2.4. From (2.1) we actually see that the right hand side of the equality consists only of odd functions. This makes sense that we only require odd functions to approximate an odd function.

Remark 2.5. (2.1) is interesting in itself. Consider if we were to plug in the value $x = \pi/2$. Then, it is clear that for all even n, $\sin(n\pi/2) = 0$. Thus, let $n = 2k + 1$, an odd integer. Since $n + 1$ will be even, $(-1)^{n+1} = 1$, and for $n = 2k + 1$, $\sin((2k + 1)\pi/2) = (-1)^k$. Using all of this information and plugging into (2.1), we now find:

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

And we see that the proof of Theorem 2.3 actually gives us two different expansions for the price of 1.

3. Application of Fourier Series: Solution of the 1-d Heat Equation

As seen in the previous section, Fourier series are very elegant and can provide some pretty interesting results. However, it should be noted that Fourier series were not motivated solely by the solution of miscellaneous infinite series. In fact, Fourier series were originally motivated by a very physical problem: the modelling of heat flow.

Definition 3.1 (The Heat Equation)**.** Let $u(x,t)$ represent the temperature in a long, thin rod oriented along the x-axis at position x and time t. For simplicity, we impose several conditions. First, assume that the rod is completely insulated, so that no heat can leave nor enter the rod through its sides. Second, assume that the heat energy is neither created nor destroyed. Then, $u(x,t)$ satisfies:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

We now derive the solution of the heat equation.

3.1. **Solution of** (3.1)**.**

Consider a rod of length l as given in definition 3.1. Let $u(x,t)$ be defined as above, and suppose it satisfies the following boundary conditions.

$$u(0,t) = u(l,t) = 0 \tag{3.2}$$

Now suppose as well that we have an initial condition given by:

$$u(x,0) = f(x) \tag{3.3}$$

where $f \in L^2(0,l)$. We now are going to make a rather ideal assumption that will be explained in a later remark. For now, we will assume that u is separable, i.e. of the form

$$u(x,t) = T(t)X(x) \tag{3.4}$$

With (3.1) and (3.4) we can now reduce this problem to a system of ordinary differential equations as so:

$$\alpha^2 X''(x)T(t) = T'(t)X(x) \implies \frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = -\lambda^2 \tag{3.5}$$

Note that in (3.5) we recognize the fact that since both sides are dependent only on x and t, they must be equal to a constant. We denote that constant by $-\lambda^2$, where $\lambda > 0$. The reason we assume this form has to do with the properties of our boundary conditions. Namely, any nonnegative constant will only yield trivial solutions. We proceed as so, and see that (3.5) has spawned two simple ODE's:

$$X''(x) = -\lambda^2 X(x)$$
$$T'(t) = -\lambda^2\alpha^2 T(t)$$

These can both be easily solved:

$$X(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x) \tag{3.6}$$

$$T(t) = ae^{-\lambda^2\alpha^2 t} \tag{3.7}$$

where c_1, c_2, and a are arbitrary constants. We now impose (3.2) onto (3.6) because the boundary conditions will have no effect on (3.7).

$$X(0) = 0 = c_2$$

$$X(l) = 0 \implies c_1 \sin(\lambda l) = 0$$

We assume that $c_1 \neq 0$, else we would have a trivial solution. Then we see:

$$\sin(\lambda l) = 0 \implies \lambda = \frac{k\pi}{l} \tag{3.8}$$

where $k \in \mathbb{Z}^+$. We now observe an interesting phenomenon. Since k is an arbitrary positive integer, we have an infinite amount of solutions. This is is recognition that the heat equation is a linear homogeneous partial differential equation (PDE). Letting $L = \alpha^2\partial_x^2 - \partial_t$, L is an operator, and u_k be an infinite sequence of solutions to this PDE, so $L(u_k) = 0$. We then see:

$$L\Big(\sum_{k=1}^{\infty} c_k u_k\Big) = \sum_{k=1}^{\infty} c_k L(u_k) = 0$$

where c_k are arbitrary constants. We thus see that any linear combination of these solutions is also a solution.

We then have:

$$u(x,t) = \sum_{k=1}^{\infty} c_k \sin\big(\frac{k\pi x}{l}\big) e^{-\alpha^2 k^2 \pi^2 t/l} \tag{3.9}$$

Where $c_k = ac_1$. It is here that the application of Fourier series comes into play. If we did not have the theory of Fourier series at hand, we would be at a dead end. However, we can now impose our initial condition given by (3.3).

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} c_k \sin\big(\frac{k\pi x}{l}\big)$$

But this is precisely a Fourier series, where the coefficients for our cosine term all vanish! Thus we can use Theorem 2.1. We set $b_n = c_k$, and see:

$$c_k = \frac{2}{l}\int_0^l f(x)\sin\big(\frac{k\pi x}{l}\big)dx \tag{3.10}$$

We now have a solution of the heat equation. Combining (3.9) and (3.10), we have:

$$u(x,t) = \sum_{k=1}^{\infty} \frac{2}{l}\left[\int_0^l f(x) sin\big(\frac{k\pi x}{l}\big)dx\right] \sin\big(\frac{k\pi x}{l}\big)e^{-\alpha^2 k^2 \pi^2 t/l}$$

Remark 3.2. The assumption (3.4) actually has a very rigorous foundation. In general, we can make this assumption if the associated Hilbert Space $\mathcal{H}$ is separable. Separability implies that there is a countable, dense basis for $\mathcal{H}$. This theory is beyond the scope of the paper, but is very interesting in itself.

4. Hilbert Spaces and Generalized Fourier Series

We now present an extremely elementary discussion of the theory of orthogonality and Hilbert Spaces.

Definition 4.1 (Hilbert Space)**.** A Hilbert Space $\mathcal{H}$ is an inner product space where every convergent Cauchy sequence converges to an element in $\mathcal{H}$ (i.e a complete inner product space).

The reader is assumed familiar with the axioms of inner products and conditions of orthogonality. We can now present some of the abstract theory behind generalized Fourier Series and deduce the trigonometric case given by Theorem 2.1.

Theorem 4.2. *For $1 \leq p \leq \infty$, the space L^p is complete.*

The proof for this is well beyond the scope of the paper, and employs many advanced techniques from the theory of real analysis. We will, however, use this theorem for a quick corollary.

Corollary. $L^2(a,b)$ endowed with the inner product $\int_a^b f(x)g(x)dx$ is a Hilbert space.

Proof. Since L^2 is complete by theorem 4.2, it suffices to check that $\int_a^b f(x)g(x)dx$ satisfies the axioms of an inner product. Since we are assuming f and g are real functions, we only consider the inner product axioms for a real vector space. Symmetry and bilinearity are clear. For positive semi-definiteness, let $f \in L^2(a,b)$. Then, as is easy to prove, $\int_a^b f^2(x)dx = 0$ iff $f(x) = 0$ almost everywhere, since $f^2(x)$ is clearly nonnegative. $\square$

For the next theorem, it is standard that a basis in the language of Hilbert spaces is defined to be a maximal orthonormal set.

Definition 4.3. The set $\{\phi_n\}$ of vectors are called a maximal orthonormal set if $\langle \phi_j, \phi_k \rangle = 0$ when $k \neq j$, $\langle \phi_j, \phi_j \rangle = 1$, and if $\psi \notin \{\phi_n\}$ and $\langle \psi, \phi_j \rangle = 0$, then $\psi = 0$.

Theorem 4.4 (Generalized Fourier Series)**.** *Let $\mathcal{H}$ be a Hilbert space. If $\{\phi_k\}$ is a basis for $\mathcal{H}$, and $f \in \mathcal{H}$, then:*

$$f = \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i \tag{4.1}$$

where n is the dimension of $\mathcal{H}$.

Proof. By means of Zorn's Lemma, it is well known that every vector space has a basis. By the Grahm-Schmidt process, every basis can be made orthonormal. We thus assume that $\mathcal{H}$ has a basis as defined by definition 4.3. Let $f \in \mathcal{H}$. Then $f \in \text{Span}\{\phi_k\}$, since this is an independent set. Thus, by definition,

$$f = \sum_{i=1}^{n} c_i \phi_i$$

for some constants c_i. Now take the inner product of both sides with any ϕ_i. We have:

$$\langle \phi_i, f \rangle = c_i$$

and the result (4.1) follows. □

Corollary. Any function $f \in L^2(a,b)$ can be given as in Theorem 2.1.

Proof. It can be easily shown that

$$\left\{ \cos\left(\frac{2n\pi x}{b-a}\right) \right\}_{n=1}^{\infty}$$

and

$$\left\{ \sin\left(\frac{2n\pi x}{b-a}\right) \right\}_{n=1}^{\infty}$$

are a basis for $L^2(a,b)$ if you take the union of these two sets under the inner product given by

$$\langle f, g \rangle = \frac{2}{b-a} \int_a^b f(x) g(x) dx$$

Thus, we just use Theorem 4.4 and the result is obvious. □

Theorem 4.5 (Parseval's Equality)**.** *Let $\mathcal{H}$ be a Hilbert space and $f \in \mathcal{H}$. Then:*

$$\langle f, f \rangle = \sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2 \tag{4.2}$$

where n is the dimension of $\mathcal{H}$.

Proof. The proof is obvious after taking the inner product of both sides of (4.1) with themselves. ☐

Remark 4.6. As in the proof for theorem 2.1, we can define the exact same inner product and the same basis and easily derive Lemma 2.2

5. Conclusion

We started with a very simple question and noticed an interesting pattern. Namely, how π seems to pop up in the solution of numerous infinite series. We then tried to explain this pattern by means of the trigonometric Fourier series, and derived two very interesting and rather nontrivial results. This then led us to examine how Fourier series can be used in mathematical physics, and we were able to solve the 1-dimensional heat equation. Finally, in order to see the abstract theory working under the hood of the trigonometric Fourier series, we were led to a more general discussion of some of the properties of Hilbert spaces and maximal orthonormal sets.

References

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E-mail address: kv00767@georgiasouthern.edu

E-mail address: cj08188@georgiasouthern.edu