AN EXPLICIT SEMI-FACTORIAL COMPACTIFICATION OF THE NÉRON MODEL

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ABSTRACT. C. Pépin recently constructed a semi-factorial compactification of the Néron model of an abelian variety using the flattening technique of Raynaud–Gruson. Here we prove that an explicit semi-factorial compactification is a certain moduli space of sheaves — the family of compactified jacobians.

Here we prove that the family of compactified jacobians is a semi-factorial compactification of the Néron model of the jacobian. Semi-factoriality is a weakening of factoriality, the condition that the local rings are unique factorization domains. In [Pép13], Pépin introduced the condition and proved that the Néron model of an abelian variety $A_K$ over the field of fractions $K$ of a discrete valuation ring $R$ admits a semi-factorial compactification $\overline{A}$, a result which he used in [Pép12] to study the Néron pairing and algebraic equivalence.

Pépin constructed the compactification using the flattening technique of Raynaud–Gruson [RG71]. We give an alternative construction when $A_K = J_K$ is a jacobian satisfying suitable hypotheses. We prove that an explicit semi-factorial compactification is given by a compactification of $J_K$ as a moduli space — by the family of compactified jacobians.

What is the compactified jacobian? Suppose $A_K = J_K$ is the jacobian of the smooth curve $X_K$. The curve $X_K$ extends to a regular model $X/S$ over $S = \text{Spec}(R)$. The jacobian $J_K$ is the moduli space of degree 0 line bundles on $X_K$, and we can try to extend it to a family $J/S$ by adding over the point $0 \in S$ a moduli space of sheaves on $X_0$. When $X_0$ is geometrically integral, we can extend $J_K$ by adding the moduli space of degree 0 rank 1, torsion-free sheaves on $X_0$, and this extended family is the family of compactified jacobians. This extended family is badly behaved when $X_0$ fails to be geometrically integral; a well-behaved space can be recovered by imposing e.g. a stability condition, but we do not work with these more general spaces in this paper.

The line bundle locus $J/S$ in a family of compactified jacobians $\overline{J}/S$ is canonically isomorphic to the Néron model of $J_K$. Indeed, this is (a special case of) [Kas13, Theorem 3.9], a result that extends earlier work on the topic [OS79, Bus08, Cap08a, Cap08b, Cap12, MV12]. Compactified jacobians are proper by construction, so $\overline{J}/S$ is a compactification of the Néron model. When the Picard rank of $J_K$ is 1, $\overline{J}/S$ has the desirable properties studied by Pépin:

**Main Theorem.** The Altman–D’Souza–Kleiman family of compactified jacobians $\overline{J}/S$ is a semi-factorial model of the Néron model provided the Picard rank of $J_K$ is 1.

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The Main Theorem includes the explicit hypothesis that $J_K$ has Picard rank 1 and the implicit hypothesis that the special fiber $X_0$ is geometrically integral. How are these hypotheses used? When the hypothesis that $X_0$ is geometrically integral fails, the Altman–D’Souza–Kleiman family $\overline{J}/S$ is not defined, but it is possible to define a more general family $\overline{J}/S$. The proof we give here does not immediately apply to these more general families. In proving the Main Theorem, we use the fact that translation $\tau_a : J_K \rightarrow J_K$ by a point $a \in J_K(K)$ extends to an automorphism $\overline{J} \rightarrow \overline{J}$. It is not known if more general $\overline{J}$'s have this extension property; the issue is that, when $X_0$ is reducible, the tensor product of two slope semi-stable line bundles can fail to be semi-stable.

The hypothesis that $J_K$ has Picard rank 1 is used to assert that the Néron–Severi group $\text{NS}(J_K)$ is generated by classes that extend to $\overline{J}$. Under the rank 1 hypothesis, $\text{NS}(J_K)$ is generated by the class of the theta divisor, and Esteves and Soucaris have (independently) shown that this divisor extends. In general, when $\text{NS}(J_K)$ is generated by classes that extend, our proof shows that $J_K$ is semi-factorial, and it would be desirable to have more general results describing when classes in $\text{NS}(J_K)$ extend.

I. Preliminaries. Here we collect results from the literature. Fix a discrete valuation ring (or dvr for short) $R$ with field of fractions $K$ and residue field $k(0)$. Set $S = \text{Spec}(R)$ and $0 = \text{Spec}(k(0))$. We fix a smooth curve $X_K/\text{Spec}(K)$ (i.e. a $K$-scheme of pure dimension 1 that is proper, smooth, and geometrically connected over $K$) that we assume has genus $g \geq 1$ and study the associated jacobian $J_K/\text{Spec}(K)$. The jacobian is a $g$-dimensional abelian variety that represents the étale sheaf parameterizing degree 0 line bundles on $X_K$, and it extends to the Néron model $J/S$, a certain (possibly nonproper) $S$-scheme. We omit the definition, but one consequence, which we will use, is that the restriction map $J(S) \rightarrow J_K(K)$ is surjective, i.e. the weak Néron Mapping Property holds.

To study compactifications of $J_K$, we make the following definitions.

**Definition 1.** A $S$-scheme $V/S$ is **semi-factorial** if the restriction map

1. $\text{Pic}(V) \rightarrow \text{Pic}(V_K)$

on Picard groups is surjective.

If $V/S$ is separated and of finite type over $S$, then an $S$-**compactification** of $V/S$ is a proper $S$-scheme $\overline{V}/S$ and a $S$-immersion $V \rightarrow \overline{V}$ with dense image. An $S$-compactification is a **semi-factorial model** if $\overline{V}/S$ is flat and projective over $S$, normal, and semi-factorial. A semi-factorial model is a **regular model** if $\overline{V}$ is a regular scheme.

Corollaire 6.4 of [Pép13] states that the Néron model $J/S$ admits a semi-factorial model. In fact, the Corollaire states that the semi-factorial model can be chosen to have certain desirable base-change properties, which we discuss in Remark 5.

The curve $X_K$ admits a regular model $X/S$ because resolution of singularities holds for arithmetic surfaces [Lip78]. (Lipman’s result is stated for $R$ excellent, but the argument on [DM69, page 87] shows that this hypothesis can be removed.) For the remainder of this paper, we fix a regular model $X$ satisfying
Assumption. \( X/S \) is a regular model of \( X_K \) with geometrically integral special fiber.

With this assumption, the Altman–D’Souza–Kleiman family of compactified jacobians \( \overline{J}/S \) associated to \( X/S \) is defined. The family of compactified jacobians is an \( S \)-scheme \( \overline{J}/S \) that is projective over \( S \) and represents the étale sheaf parameterizing families of degree 0 rank 1, torsion-free sheaves on \( X/S \) [AK80, (8.10) Theorem]. (Under more restrictive hypotheses, this is [D’S79, Theorem II.4.1].) The line bundle locus in \( \overline{J} \) is an open subscheme \( J \) that is the Néron model of \( J_K \) [Kas13, Theorem 3.9].

We use the Picard scheme of \( J_K \) in proving the Main Theorem. The Picard scheme \( \text{Pic}(J_K/K)/\text{Spec}(K) \) is a \( K \)-group scheme that is locally of finite type over \( K \) and represents the étale sheaf parameterizing line bundles on \( J_K \). The line bundles that are algebraically equivalent to zero are parameterized by the identity component \( \text{Pic}^0(J_K/K) \) of the Picard scheme, which is an open and closed \( K \)-subgroup scheme that is of finite type over \( K \).

Algebraic equivalence classes of line bundles form the Néron–Severi group which is defined as

\[
\text{NS}(J_K) := \frac{\text{Pic}(J_K/K)(\overline{K})}{\text{Pic}^0(J_K/K)(\overline{K})}
\]

for \( \overline{K} \) a fixed algebraic closure of \( K \). This group is finitely generated, hence has a well-defined rank called the Picard rank.

The Picard rank of \( J_K \) is at least 1. Indeed, \( J_K \) admits a special type of divisor: the classical theta divisor. If \( N_K \) is a line bundle of degree \( g - 1 \) on \( X_K \), then

\[
\Theta_K := \{ [L_K] : h^0(X_K, L_K \otimes N_K) \neq 0 \} \subset J_K
\]

is an ample divisor that defines a principal polarization. That is, the homomorphism

\[
\phi : J_K \to \text{Pic}^0(J_K/\overline{K}) \text{ defined by } \phi(a) = \mathcal{O}_K(\tau^*_a(\Theta_K) - \Theta_K)
\]

is an isomorphism. Here \( \tau_a \) is the translation-by-\( a \) map.

The divisor \( \Theta_K \) depends on the choice of \( N_K \), but its image in the Néron–Severi group is independent of the choice, and we denote this common image by \( \theta \). Because \( \Theta_K \) is a principal polarization, \( \theta \) is nonzero, and furthermore:

**Lemma 2.** The class \( \theta \) freely generates \( \text{NS}(J_K) \) when the Picard rank of \( J_K \) is 1.

**Proof.** If \( J_K \) has Picard rank 1, then the Néron–Severi group \( \text{NS}(J_K) \) is cyclic because it is torsion-free [Mum70, Corollary 2, page 178], so we may fix a generator \( c \). Writing \( \theta = n \cdot c \) for some \( n \in \mathbb{Z} \), we have

\[
n^g \cdot (c^g/g!) = \theta^g/g! = 1 \text{ by the Riemann–Roch Formula.}
\]

So \( n^g \) divides 1 and hence \( n = \pm 1 \). \( \square \)
II. Proof of the Main Theorem. Here we prove that $J/S$ is a semi-factorial model of the Néron model provided the Picard rank of $J_K$ is 1.

**Lemma 3.** $J \to S$ is flat, and $J$ is Cohen–Macaulay and normal.

*Proof.* Theorem (9) of [AIK77] states that $J \to S$ is flat with Cohen–Macaulay fibers. (That theorem includes the hypothesis that $X$ lies on a $S$-smooth family of surfaces, but we can reduce to this case by arguing as in the proof of [EGK02, Lemma 3.4].) Since $S$ is Cohen–Macaulay, we can conclude that $J$ itself is Cohen–Macaulay.

We prove $J$ is normal using Serre’s criteria. To verify the criteria, we need to show that Condition R1 holds. The line bundle locus $J_0 \subset J$ is dense in the special fiber by [AIK77, Theorem (9)], so the line bundle locus $J \subset J$ in the total space contains all codimension 1 points. The locus $J$ is contained in the smooth locus of $J/S$, hence in the regular locus of $J$, and so Condition R1 is satisfied. $\square$

**Proof of the Main Theorem.** By Lemma 3 we just need to show that $J/S$ is semi-factorial, i.e. $(3)$ $\text{Pic}(J) \to \text{Pic}(J_K)$ is surjective.

First, assume that $X$ admits a line bundle $N$ with fiber-wise degree $g-1$. Then the set $\{[\mathcal{L}] \in \tilde{J}: h^0(X, \mathcal{L} \otimes N) \neq 0\} \subset \tilde{J}$ is the support of a relatively effective divisor $\Theta$ that extends the classical theta divisor by [Sou94, Theorem 13] (or [Est97, page 184]). In particular, $\mathcal{O}_{J_K}(\Theta_K)$ lies in the image of (3).

That image also contains all line bundles algebraically equivalent to zero. Indeed, the polarization isomorphism $\phi$ from Equation (2) is defined over $K$, so if $\mathcal{L}_K$ is a line bundle on $J_K$ that is algebraically equivalent to zero, then we can write $[\mathcal{L}_K] = \phi(\alpha_K)$ for some $\alpha_K \in J_K(K)$. Here $[\mathcal{L}_K] \in \text{Pic}^0(J_K/K)(K)$ is the point represented by $\mathcal{L}_K$. The $S$-scheme $J/S$ satisfies the Néron Mapping Property (by e.g. [Kas13, Theorem 3.9]), so $\alpha_K \in J_K(K)$ is the restriction of some $\alpha \in J(S)$. The line bundle locus $J$ acts on $J$ (by tensor product), so translation $\tau_\alpha: \tilde{J} \to \tilde{J}$ by $\alpha$ is well-defined, and the line bundle $\mathcal{L} := \mathcal{O}_{J}(\tau_\alpha(\Theta) - \Theta)$ extends $\mathcal{L}_K$.

We have now shown that the image of (3) contains both $\mathcal{O}_{J_K}(\Theta_K)$ and the line bundles algebraically equivalent to zero. Together these line bundles generate $\text{Pic}(J_K)$ by Lemma 2, so (3) is surjective, proving the theorem in the special case that a $N$ exists.

In the general case, we argue as allows. Given a line bundle $\mathcal{L}_K$ on $J_K$, we can extend $\mathcal{L}_K$ to a family $\mathcal{L}$ of rank 1, torsion-free sheaves on $J$ (by e.g. the $S$-projectivity of the relevant compactified Picard scheme). There exists a line bundle $N$ with fiber-wise degree $g-1$ on $X_T$ for some étale cover $T \to S$ with $T$ the spectrum of a dvr because $X_0$ is geometrically reduced. Say $L$ is the field of fractions of the dvr $\Gamma(T, \mathcal{O}_T)$. The base-change $X_T$ remains regular, so $\mathcal{L}_T$ extends to a line bundle on $\tilde{J}_T$. This extension must equal $\mathcal{L}_T$ (by e.g. the $S$-separatedness of the relevant compactified Picard scheme), so $\mathcal{L}_T$ and hence $\mathcal{L}$ must be a line bundle. $\square$
Remark 4. Does \( J \) satisfy stronger conditions than semi-factoriality? Typically \( J \) does not satisfy the condition of regularity. Let \( K = \mathbb{Q}, R = \mathbb{Z}[\alpha] \) (the localization of \( \mathbb{Z} \) at 3), \( S = \text{Spec}(R) \), and \( X/S \) the minimal proper regular model of the affine curve \( \text{Spec}(R[x, y]/(y^2 - x^2(x - 1)^2(x^2 + 1) - 3)) \). The family \( X/S \) is a family of genus 2 curves with special fiber \( X_0 \) a rational curve with 2 nodes. Consider the family of compactified jacobians \( J/S \) associated to \( X/S \).

If \( \nu: \mathbb{P}^1 \cong \widetilde{X}_0 \to X_0 \) is the normalization, then \( J \) has a singularity at the rank 1, torsion-free sheaf \( I := \nu_*\mathcal{O}(-2) \). The singularity of \( J \) at 1 is computed in [Kas09]. The sheaf \( I \) fails to be locally free at 2 nodes, so by [Kas09, Lemma 6.2] the completed local ring is isomorphic to

\[
\hat{O}_{T, \frak{m}} = \hat{R}[[a_1, a_2, b_1]]/(a_1b_1 - 3, a_2b_2 - 3)
\]

This ring not only fails to be regular, but it also fails to be factorial. (The height 1 prime \((a_1, a_2)\) is nonprincipal because the images of \(a_1, a_2\) in the quotient \((3, a_i, b_i) \sim (3, a_i, b_i)^2\) are linearly independent.)

However, \( J/S \) is semi-factorial. Indeed, by the Main Theorem, we just need to show that \( J_k = J_0 \) has Picard rank 1, and we do so as follows. The Néron–Severi group \( \text{NS}(J_\mathbb{Q}) \) injects into the endomorphism ring \( \text{End}(J_\mathbb{Q}) \), and we compute this endomorphism ring by relating it to the endomorphism ring of the reduction of \( J_\mathbb{Q} \) at a prime of good reduction.

Both the curve \( X_\mathbb{Q} \) and its jacobian \( J_\mathbb{Q} \) have good reduction at the primes \( p = 5, 13 \), as can be seen by reducing the equation \( y^2 = x^2(x - 1)^2(x^2 + 1) + 3 \mod p \). Using this equation to naively count \( \mathbb{F}_p^{n^2} \)-points, we compute that the characteristic polynomial \( f_p \) of the Frobenius endomorphism of \( J_{\mathbb{F}_p} \) is

\[
\begin{align*}
f_5 &= x^4 - 2x^3 + 3x^2 - 10x + 25, \\
f_{13} &= x^4 + 7x^3 + 35x^2 + 91x + 169.
\end{align*}
\]

Applying [HZ02, Theorem 6] to these polynomials, we get that the reduction \( J_{\mathbb{F}_p} \) is absolutely simple for \( p = 5, 13 \), so \( \mathbb{Q} \otimes \text{End}(J_{\mathbb{F}_p}) = \mathbb{Q}[x]/(f_p) \).

The reduction map injects \( \mathbb{Q} \otimes \text{End}(J_\mathbb{Q}) \) into \( \mathbb{Q} \otimes \text{End}(J_{\mathbb{F}_p}) \) for \( p = 5, 13 \). A computation shows that the discriminant of \( \mathbb{Q}[x]/(f_5) \) is coprime to the discriminant of \( \mathbb{Q}[x]/(f_{13}) \), and \( \mathbb{Q} \) has no nontrivial unramified extensions, so the only field contained in both \( \mathbb{Q}[x]/(f_5) \) and \( \mathbb{Q}[x]/(f_{13}) \) is \( \mathbb{Q} \). In particular, \( \text{End}(J_\mathbb{Q}) = \mathbb{Z} \). This example was suggested to the author by Bjorn Poonen.

Remark 5. Corollaire 6.4 of [Pép13] proves that a semi-factorial model \( J/S \) of \( J_k \) can be chosen to be well-behaved with respect to certain dvr extensions. To be precise, given morphisms \( T_i \to S, \ldots, T_n \to S \) corresponding to extensions of \( R \) contained in the strict henselization \( R^\text{hs} \), a semi-factorial model \( J/S \) can be chosen so that \( \widetilde{J}/T \) is a semi-factorial model when \( T \to S \) equals either some \( T_i \to S \) or a morphism corresponding to a “permise” dvr extension.

The family \( J/S \) of compactified jacobians satisfies this condition. In fact, it satisfies a stronger condition. By definition the formation of the family of compactified jacobians commutes with arbitrary base change, so if \( T \to S \) is a morphism that corresponds to a dvr
extension, then $\tilde{J}_T/T$ is a semi-factorial model of the Néron model provided $X_T$ is regular. The scheme $X_T$ is regular when $T \to S$ is one of the morphisms considered by Pépin or more generally when $T \to S$ is regular and surjective (see [Pép13, Remarque 5.5]).

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TO BE ADDED.

REFERENCES

[Mum70] D. Mumford, Abelian varieties, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970. MR 0282985 (44 #219)


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