§7.4.

P2. Solution:

$$(f*g)(t) = t * e^{at}$$
$$= \int_0^t \tau e^{a(t-\tau)} d\tau$$
$$= e^{at} \int_0^t \tau e^{-a\tau} d\tau$$
$$= e^{at}\left[\tau\left(\frac{e^{-a\tau}}{-a}\right)\Big|_{\tau=0}^{t} - \int_0^t \left(\frac{e^{-a\tau}}{-a}\right) d\tau\right]$$
$$= e^{at}\left[-\frac{te^{-at}}{a} - \left[\frac{e^{-a\tau}}{a^2}\right]_{\tau=0}^{t}\right]$$
$$= e^{at}\left[-\frac{te^{-at}}{a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right]$$
$$= -\frac{t}{a} - \frac{1}{a^2} + \frac{e^{at}}{a^2}$$

P3. Solution:

$$(f*g)(t) = \sin t * \sin t$$
$$= \int_0^t \sin\tau \sin(t-\tau) d\tau$$
$$= \int_0^t \frac{1}{2}[\cos(2\tau - t) - \cos t] d\tau$$
$$= \frac{1}{2}\left[\frac{1}{2}\sin(2\tau - t) - \tau\cos t\right]_{\tau=0}^{t}$$
$$= \frac{1}{2}\left(\frac{1}{2}\sin t + \frac{1}{2}\sin t - t\cos t\right)$$
$$= \frac{1}{2}(\sin t - t\cos t)$$

P4. Solution:

$$(f*g)(t) = t^2 * \cos t$$
$$= \int_0^t \tau^2 \cos(t-\tau) d\tau$$
$$= \tau^2(-\sin(t-\tau))\Big|_{\tau=0}^{t} - \int_0^t 2\tau(-\sin(t-\tau)) d\tau$$
$$= \int_0^t 2\tau \sin(t-\tau) d\tau$$
$$= 2\tau\cos(t-\tau)\Big|_{\tau=0}^{t} - \int_0^t 2\cos(t-\tau) d\tau$$
$$= 2t - 2\left[-\sin(t-\tau)\right]_{\tau=0}^{t}$$
$$= 2t - 2\sin t$$

P7. Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$
$$= 1 * e^{3t}$$
$$= \int_0^t e^{3(t-\tau)} d\tau$$
$$= e^{3t}\left[\frac{e^{-3t}}{-3} + \frac{1}{3}\right]$$
$$= -\frac{1}{3} + \frac{e^{3t}}{3}$$

P8. Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$
$$= 1 * \left(\frac{1}{2}\sin 2t\right)$$
$$= \int_0^t \frac{1}{2}\sin 2(t-\tau) d\tau$$
$$= \frac{1}{2}\left[\frac{1}{2}\cos 2(t-\tau)\right]_{\tau=0}^{t}$$
$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\cos 2t\right]$$
$$= \frac{1}{4} - \frac{1}{4}\cos 2t$$

P11. Solution:

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$
$$= \cos 2t * \cos 2t$$
$$= \int_0^t \cos 2\tau \cos 2(t-\tau) d\tau$$
$$= \int_0^t \frac{1}{2}[\cos(2t - 4\tau) + \cos 2t] d\tau$$
$$= \frac{1}{4}\sin 2t + \frac{1}{2}t\cos 2t$$

P15. Solution:

$$\mathcal{L}\{t\sin 3t\} = -\frac{d}{ds}[\mathcal{L}\{\sin 3t\}]$$
$$= -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$
$$= \frac{6s}{(s^2+9)^2}, \quad s > 0$$

P16. Solution:

$$\mathcal{L}\{t^2\cos 2t\} = \frac{d^2}{ds^2}[\mathcal{L}\{\cos 2t\}]$$
$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$
$$= \frac{d}{ds}\left[\frac{(s^2+4)-s(2s)}{(s^2+4)^2}\right]$$
$$= \frac{d}{ds}\left[\frac{-s^2+4}{(s^2+4)^2}\right]$$
$$= \frac{2s^3-24s}{(s^2+4)^3}, \quad s>0$$

P19. Solution:

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^{+\infty} \mathcal{L}\{\sin t\}\, d\sigma$$
$$= \int_s^{+\infty} \frac{1}{\sigma^2+1}\, d\sigma$$
$$= \left[\arctan \sigma\right]_{\sigma=s}^{+\infty}$$
$$= \frac{\pi}{2} - \arctan s$$

P23. Solution:

$$\mathcal{L}^{-1}\left\{\ln\frac{s-2}{s+2}\right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln\frac{s-2}{s+2}\right]\right\}$$
$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s+2}\right\}$$
$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{4}{s^2-4}\right\}$$
$$= -\frac{1}{t}\cdot(2\sinh 2t)$$
$$= -\frac{2\sinh 2t}{t}$$

P25. Solution:

$$\mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{(s+2)(s-3)}\right\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln\frac{s^2+1}{(s+2)(s-3)}\right]\right\}$$
$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s+2} - \frac{1}{s-3}\right\}$$
$$= -\frac{1}{t}\left(2\cos t - e^{-2t} - e^{3t}\right)$$

P29. Solution:

$$tx'' + (t-2)x' + x = 0; \quad x(0) = 0$$
$$\mathcal{L}\{tx'' + (t-2)x' + x\} = \mathcal{L}\{0\}.$$

let $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x'(t)\} = sX(s)$$
$$\mathcal{L}\{x''(t)\} = s^2X(s) \qquad \text{let } x'(0) = 0$$

$$\Rightarrow \mathcal{L}\{(t-2)x'\} = \mathcal{L}\{tx'\} - 2\mathcal{L}\{x'\}$$
$$= -\frac{d}{ds}[sX(s)] - 2sX(s)$$
$$= -X(s) - sX'(s) - 2sX(s)$$

$$\mathcal{L}\{tx''\} = -\frac{d}{ds}[s^2X(s)]$$
$$= -2sX(s) - s^2X'(s)$$

$$\Rightarrow [-2sX(s) - s^2X'(s)] + [-\cancel{X(s)} - sX'(s) - 2sX(s)] + \cancel{X(s)} = 0$$

$$\Rightarrow \quad -(s+1)X'(s) - 4X(s) = 0$$
$$\frac{X'(s)}{X(s)} = -\frac{4}{s+1}$$
$$\Rightarrow X(s) = C(s+1)^{-4}$$
$$\Rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{C(s+1)^{-4}\}$$
$$= Ce^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$
$$= Ce^{-t}t^3$$