§7.3

P1. Solution: $\mathcal{L}\{t^4\} = \frac{24}{s^5}$, $s>0$.

$\Rightarrow \mathcal{L}\{t^4 e^{\pi t}\} = \frac{24}{(s-\pi)^5}$, $s>\pi$

P3. Solution: $\mathcal{L}\{\sin 3\pi t\} = \frac{3\pi}{s^2+(3\pi)^2}$

$= \frac{3\pi}{s^2+9\pi^2}$, $s>0$

$\Rightarrow \mathcal{L}\{e^{-2t}\sin 3\pi \pi\} = \frac{3\pi}{(s+2)^2+9\pi^2}$

P5. Solution:

$\frac{3}{2s-4} = \frac{3}{2(s-2)}$

$\Rightarrow \mathcal{L}^{-1}\{\frac{3}{2s-4}\} = \frac{3}{2}\mathcal{L}^{-1}\{\frac{1}{(s-2)}\}$

$= \frac{3}{2}e^{2t}\mathcal{L}^{-1}\{\frac{1}{s}\}$

$= \frac{3}{2}e^{2t}$

P7. Solution:

$\frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$

$\Rightarrow \mathcal{L}^{-1}\{\frac{1}{s^2+4s+4}\} = \mathcal{L}^{-1}\{\frac{1}{(s+2)^2}\}$

$= e^{-2t}\mathcal{L}^{-1}\{\frac{1}{s^2}\}$

$= te^{-2t}$

P8. Solution:

$\frac{s+2}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1}$

$\Rightarrow \mathcal{L}^{-1}\{\frac{s+2}{s^2+4s+5}\} = \mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+1}\}$

$= e^{-2t}\mathcal{L}^{-1}\{\frac{s}{s^2+1}\}$

$= e^{-2t}\cos t$

P11. Solution: $F(s) = \frac{1}{s^2-4} = \frac{A}{s-2}+\frac{B}{s+2}$

$= \frac{A(s+2)+B(s-2)}{s^2-4}$

$= \frac{(A+B)s+(2A-2B)}{s^2-4}$

$\Rightarrow \begin{array}{l} A+B=0 \\ 2A-2B=1 \end{array} \Rightarrow \begin{array}{l} A=\frac{1}{4} \\ B=-\frac{1}{4} \end{array}$

$\Rightarrow \mathcal{L}^{-1}\{\frac{1}{s^2-4}\} = \mathcal{L}^{-1}\{\frac{1/4}{s-2}\}+\mathcal{L}^{-1}\{\frac{(-1/4)}{s+2}\}$

$= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$

$= \frac{1}{2}\left(\frac{e^{2t}-e^{-2t}}{2}\right)$

$= \frac{1}{2}\sinh 2t$

P13. Solution:

$F(s) = \frac{5-2s}{s^2+7s+10} = \frac{5-2s}{(s+2)(s+5)}$

$= \frac{A}{s+2}+\frac{B}{s+5} = \frac{A(s+5)+B(s+2)}{(s+2)(s+5)}$

$= \frac{(A+B)s+(5A+2B)}{(s+2)(s+5)}$

$\Rightarrow \left.\begin{array}{l} A+B=-2 \\ 5A+2B=5 \end{array}\right\} \Rightarrow \begin{array}{l} A=3 \\ B=-5 \end{array}$

$\Rightarrow \mathcal{L}^{-1}\{\frac{5-2s}{s^2+7s+10}\} = \mathcal{L}^{-1}\{\frac{3}{s+2}\}+\mathcal{L}^{-1}\{\frac{-5}{s+5}\}$

$= 3e^{-2t}-5e^{-5t}$

P15. Solution:

$\frac{1}{s^3-5s^2} = \frac{1}{s^2(s-5)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-5}$

$= \frac{As(s-5)+B(s-5)+Cs^2}{s^2(s-5)} = \frac{(A+C)s^2+(-5A+B)s-5B}{s^2(s-5)}$

$\Rightarrow \left.\begin{array}{l} A+C=0 \\ -5A+B=0 \\ -5B=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=-\frac{1}{25} \\ B=-\frac{1}{5} \\ C=\frac{1}{25} \end{array}$

$\Rightarrow \mathcal{L}^{-1}\{\frac{1}{s^3-5s^2}\} = \mathcal{L}^{-1}\{\frac{(-1/25)}{s}\}+\mathcal{L}^{-1}\{\frac{(-1/5)}{s^2}\}$

$+\mathcal{L}^{-1}\{\frac{1/25}{s-5}\}$

$= -\frac{1}{25}-\frac{t}{5}+\frac{1}{25}e^{5t}$

P27. Solution:

$x''+6x'+25x=0$; $x(0)=2$, $x'(0)=3$.

$\Rightarrow \mathcal{L}\{x(t)\} = X(s)$

$\mathcal{L}\{x'(t)\} = sX(s)-2$

$\mathcal{L}\{x''(t)\} = s^2X(s)-2s-3$

From $\mathcal{L}\{x''+6x'+25x\}=\mathcal{L}\{0\}$, we get

$$[s^2X(s)-2s-3]+6[sX(s)-2]+25X(s)=0$$

$$(s^2+6s+25)X(s)=2s+15$$

$$X(s)=\frac{2s+15}{s^2+6s+25}$$

$$\frac{2s+15}{s^2+6s+25}=\frac{2s+15}{(s+3)^2+16}=\frac{2(s+3)+9}{(s+3)^2+16}$$

$$=\frac{2(s+3)}{(s+3)^2+16}+\frac{9}{(s+3)^2+16}$$

$$\Rightarrow x(t)=\mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+16}\right\}+\mathcal{L}^{-1}\left\{\frac{9}{(s+3)^2+16}\right\}$$

$$=e^{-3t}\mathcal{L}^{-1}\left\{\frac{2s}{s+16}\right\}+e^{-3t}\left\{\frac{9}{s^2+16}\right\}$$

$$=e^{-3t}(2\cos 4t)+e^{-3t}\left(\frac{9}{4}\sin 4t\right)$$

$$=e^{-3t}\left(2\cos 4t+\frac{9}{4}\sin 4t\right)$$

P28. Solution:

$$x''-6x'+8x=2;\ x(0)=0=x'(0)$$

$$\Rightarrow \mathcal{L}\{x(t)\}=X(s)$$

$$\mathcal{L}\{x'(t)\}=sX(s)$$

$$\mathcal{L}\{x''(t)\}=s^2X(s)$$

$$\mathcal{L}\{x''-6x'+8x\}=\mathcal{L}\{2\}\Rightarrow$$

$$s^2X(s)-6sX(s)+8X(s)=\frac{2}{s}$$

$$\Rightarrow(s^2-6s+8)X(s)=\frac{2}{s}$$

$$X(s)=\frac{2}{s(s^2-6s+8)}$$

$$\frac{2}{s(s^2-6s+8)}=\frac{2}{s(s-2)(s-4)}=\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s-4}$$

$$=\frac{A(s-2)(s-4)+Bs(s-4)+Cs(s-2)}{s(s-2)(s-4)}$$

let $s=0$, we get $2=8A\Rightarrow A=\frac{1}{4}$

let $s=2$, we get $2=-4B\Rightarrow B=-\frac{1}{2}$

let $s=4$, we get $2=8C\Rightarrow C=\frac{1}{4}$

Thus, $x(t)=\mathcal{L}^{-1}(X(s))=\mathcal{L}^{-1}\left\{\frac{1/4}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{(-1/2)}{s-2}\right\}+\mathcal{L}^{-1}\left\{\frac{1/4}{s-4}\right\}$

$$=\frac{1}{4}-\frac{1}{2}e^{2t}+\frac{1}{4}e^{4t}$$

P29. solution:

$$x''-4x=3t;\ x(0)=x'(0)=0$$

$$\Rightarrow \mathcal{L}\{x(t)\}=X(s)$$

$$\mathcal{L}\{x''(t)\}=s^2X(s)$$

From $\mathcal{L}\{x''-4x\}=\mathcal{L}\{3t\}$, we get

$$s^2X(s)-4X(s)=\frac{3}{s^2}$$

$$X(s)=\frac{3}{s^2(s^2-4)}$$

$$\frac{3}{s^2(s^2-4)}=\frac{A}{s^2}+\frac{B}{s^2-4}=\frac{(A+B)s^2-4A}{s^2(s^2-4)}$$

$$\Rightarrow \begin{matrix}A+B=0\\ -4A=3\end{matrix}\Rightarrow\begin{matrix}A=-3/4\\ B=3/4\end{matrix}$$

Thus,

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{(-3/4)}{s^2}\right\}+\mathcal{L}^{-1}\left\{\frac{3/4}{s^2-4}\right\}$$

$$=-\frac{3}{4}t+\frac{3}{8}\sinh 2t$$