§7.2.

P1. Solution:

$x''+4x=0;\ x(0)=5,\ x'(0)=0$

$\Rightarrow \mathcal{L}\{x(t)\}=X(s)$

$\mathcal{L}\{x'(t)\}=sX(s)-5$

$\mathcal{L}\{x''(t)\}=s^2X(s)-5s$

Thus

$\mathcal{L}\{x''+4x\}=\mathcal{L}\{0\}$

$[s^2X(s)-5s]+4X(s)=0$

$(s^2+4)X(s)=5s$

$X(s)=\frac{5s}{s^2+4}$

$\Rightarrow x(t)=\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\{\frac{5s}{s^2+4}\}$

$=5\cos 2t$

P2. solution:

$x''+9x=0;\ x(0)=3,\ x'(0)=4$

$\Rightarrow \mathcal{L}\{x(t)\}=X(s)$

$\mathcal{L}\{x'(t)\}=sX(s)-3$

$\mathcal{L}\{x''(t)\}=s^2X(s)-3s-4$

Thus

$\mathcal{L}\{x''+9x\}=\mathcal{L}\{0\}$

$[s^2X(s)-3s-4]+9X(s)=0$

$(s^2+9)X(s)=3s+4$

$X(s)=\frac{3s+4}{s^2+9}$

$\Rightarrow x(t)=\mathcal{L}^{-1}\{\frac{3s+4}{s^2+9}\}$

$=\mathcal{L}^{-1}\{\frac{3s}{s^2+9}\}+\mathcal{L}^{-1}\{\frac{4}{s^2+9}\}$

$=3\cos 3t+\frac{4}{3}\sin 3t$

P3. $x''-x'-2x=0;\ x(0)=0,\ x'(0)=2.$

$\Rightarrow \mathcal{L}\{x(t)\}=X(s)$

$\mathcal{L}\{x'(t)\}=sX(s)$

$\mathcal{L}\{x''(t)\}=s^2X(s)-2$

$\mathcal{L}\{x''-x'-2x\}=\mathcal{L}\{0\}$

$[s^2X(s)-2]-sX(s)-2X(s)=0$

$(s^2-s-2)X(s)=2$

$X(s)=\frac{2}{s^2-s-2}$

$\frac{2}{s^2-s-2}=\frac{2}{(s-2)(s+1)}=\frac{A}{s-2}+\frac{B}{s+1}$

$=\frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$

let $s=-1$, we have $2=-3B \Rightarrow B=-\frac{2}{3}$

let $s=2$, we have $2=3A \Rightarrow A=\frac{2}{3}$

$\Rightarrow \frac{2}{s^2-s-2}=\frac{2/3}{s-2}+\frac{(-2/3)}{s+1}$

$\Rightarrow x(t)=\mathcal{L}^{-1}\{\frac{2}{s^2-s-2}\}=\mathcal{L}^{-1}\{\frac{2/3}{s-2}\}+\mathcal{L}^{-1}\{\frac{(-2/3)}{s+1}\}$

$=\frac{2}{3}e^{2t}-\frac{2}{3}e^{t}$

P5. Solution:

$x''+x=\sin 2t;\ x(0)=0=x'(0)$

$\Rightarrow \mathcal{L}\{x(t)\}=X(s)$

$\mathcal{L}\{x'(t)\}=sX(s)$

$\mathcal{L}\{x''(t)\}=s^2X(s)$

Thus

$\mathcal{L}\{x''+x\}=\mathcal{L}\{\sin 2t\}$

$s^2X(s)+X(s)=\frac{2}{s^2+4}$

$\Rightarrow X(s)=\frac{2}{(s^2+4)(s^2+1)}$

$\frac{2}{(s^2+4)(s^2+1)}=\frac{A}{s^2+4}+\frac{B}{s^2+1}=\frac{A(s^2+1)+B(s^2+4)}{(s^2+4)(s^2+1)}$

$=\frac{(A+B)s^2+(A+4B)}{(s^2+4)(s^2+1)}$

$\Rightarrow \begin{cases} A+B=0 \\ A+4B=2 \end{cases} \Rightarrow \begin{matrix} A=-2/3 \\ B=2/3 \end{matrix}$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)(s^2+1)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{(-2/3)}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2/3}{s^2+1}\right\}$$
$$= -\frac{1}{3}\sin 2t + \frac{2}{3}\sin t$$

P8. Solution:

$$x''+9x = 1;\ x(0) = 0 = x'(0)$$
$$\mathcal{L}\{x(t)\} = X(s)$$
$$\mathcal{L}\{x'(t)\} = sX(s)$$
$$\mathcal{L}\{x''(t)\} = s^2 X(s)$$

Thus

$$\mathcal{L}\{x''+9x\} = \mathcal{L}\{1\}$$
$$s^2 X(s) + 9X(s) = \frac{1}{s}$$
$$(s^2+9)X(s) = \frac{1}{s}$$
$$X(s) = \frac{1}{s(s^2+9)}$$
$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9)+s(Bs+C)}{s(s^2+9)}$$
$$= \frac{(A+B)s^2+Cs+9A}{s(s^2+9)}$$
$$\Rightarrow \left.\begin{array}{l} A+B=0 \\ C=0 \\ 9A=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=\frac{1}{9} \\ B=-1/9 \\ C=0 \end{array}$$
$$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/9}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{(-1/9)s}{s^2+9}\right\}$$
$$= \frac{1}{9} - \frac{1}{9}\cos 3t$$

P17. Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} d\tau$$
$$= \int_0^t e^{3\tau} d\tau$$
$$= \left[\frac{1}{3}e^{3\tau}\right]_{\tau=0}^{t}$$
$$= \frac{1}{3}e^{3t} - \frac{1}{3} = \frac{1}{3}(e^{3t}-1)$$

P19. Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} d\tau$$
$$= \int_0^t \frac{1}{2}\sin 2\tau \, d\tau$$
$$= \left[\frac{-1}{4}\cos 2\tau\right]_{\tau=0}^{t}$$
$$= \frac{-1}{4}(\cos 2t - 1)$$

P20. Solution:

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s(s^2+9)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{2}{s^2+9}\right\} + \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} d\tau$$
$$= \frac{2}{3}\sin 3t + \int_0^t \frac{1}{3}\sin 3\tau \, d\tau$$
$$= \frac{2}{3}\sin 3t + \left[-\frac{1}{9}\cos 3\tau\right]_{\tau=0}^{t}$$
$$= \frac{2}{3}\sin 3t - \frac{1}{9}(\cos 3t - 1)$$

P21. Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} d\tau$$
$$= \int_0^t \sin\tau \, d\tau$$
$$= \left[-\cos\tau\right]_{\tau=0}^{t} = -\cos t + 1$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} d\tau$$
$$= \int_0^t (-\cos\tau + 1)\, d\tau$$
$$= \left[-\sin\tau + \tau\right]_{\tau=0}^{t}$$
$$= -\sin t + t$$

P28. Solution:

Let $f(t) = t\cos kt$. then $\rightarrow f(0) = 0$

$$f'(t) = \cos kt - kt\sin kt,\quad f'(0) = 1$$
$$f''(t) = -k\sin kt - k\sin kt - k^2 t\cos kt$$
$$= -2k\sin kt - k^2 t\cos kt$$
$$\Rightarrow \mathcal{L}\{f''(t)\} = -2k\mathcal{L}\{\sin kt\} - k^2\mathcal{L}\{t\cos kt\}$$
$$s^2\mathcal{L}\{t\cos kt\} - 1 = \frac{-2k^2}{s^2+k^2} - k^2\mathcal{L}\{t\cos kt\}$$
$$(s^2+k^2)\mathcal{L}\{t\cos kt\} = \frac{s^2-k^2}{s^2+k^2}$$
$$\Rightarrow \mathcal{L}\{t\cos kt) = \frac{s-k^2}{(s^2+k^2)^2}$$