§4.2.

P2. $\begin{cases} x' = x - 2y & —① \\ y' = 2x - 3y & —② \end{cases}$

From ②

$x = \frac{1}{2}(y' + 3y)$

$x' = \frac{1}{2}(y'' + 3y')$

put them into ① to get

$\frac{1}{2}(y'' + 3y') = \frac{1}{2}(y' + 3y) - 2y$.

$\frac{1}{2}y'' + y' + \frac{1}{2}y = 0$

$y'' + 2y' + y = 0 \to (r^2 + 2r + 1 = 0)$

$\Rightarrow y(t) = (c_1 + c_2 t)e^{-t}$

$\Rightarrow y'(t) = (c_2 - c_1 - c_2 t)e^{-t}$

$\Rightarrow x(t) = \frac{1}{2}(y' + 3y)$

$= (\frac{1}{2}c_1 + c_2 + c_2 t)e^{-t}$

P3. $\begin{cases} x' = -3x + 2y & —① \\ y' = -3x + 4y & —② \end{cases}$

with $x(0) = 0$, $y(0) = 2$

From ②.

$x = \frac{1}{3}(-y' + 4y)$

$x' = \frac{1}{3}(-y'' + 4y')$

put into ①

$-\frac{1}{3}y'' + \frac{4}{3}y' = y' - 4y + 2y$.

$-\frac{1}{3}y'' + \frac{1}{3}y' + 2y = 0$

$y'' - y' - 6y = 0 \to (r^2 - r - 6 = 0)$

$\Rightarrow y(t) = c_1 e^{3t} + c_2 e^{-2t}$

$\to y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t}$

$\Rightarrow x(t) = -\frac{1}{3}y' + \frac{4}{3}y = \frac{1}{3}c_1 e^{3t} + 2c_2 e^{-2t}$

$\begin{cases} x(0) = 0 \\ y(0) = 2 \end{cases} \Rightarrow \begin{cases} \frac{1}{3}c_1 + 2c_2 = 0 \\ c_1 + c_2 = 2 \end{cases} \Rightarrow c_1 = \frac{12}{5}, \; c_2 = -\frac{2}{5}$

$\Rightarrow x(t) = \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t}$, $y(t) = \frac{12}{5}e^{3t} - \frac{2}{5}e^{-2t}$

P6. $\begin{cases} x' = x + 9y & —① \\ y' = -2x - 5y & —② \end{cases}$

with $x(0) = 3$, $y(0) = 2$.

From ②,

$x = -\frac{1}{2}y' - \frac{5}{2}y \Rightarrow x' = -\frac{1}{2}y'' - \frac{5}{2}y'$

put into ①

$-\frac{1}{2}y'' - \frac{5}{2}y' = -\frac{1}{2}y' - \frac{5}{2}y + 9y$

$-\frac{1}{2}y'' - 2y' - \frac{13}{2}y = 0$

$y'' + 4y' + 13y = 0 \to (r^2 + 4r + 13 = 0)$

$\Rightarrow y(t) = e^{-2t}(c_1\cos 3t + c_2 \sin 3t)$

$y'(t) = e^{-2t}[(-2c_1 + 3c_2)\cos 3t$
$+ (-3c_1 - 2c_2)\sin 3t]$

$\Rightarrow x(t) = -\frac{1}{2}y' - \frac{5}{2}y$

$= e^{-2t}[(-\frac{3}{2}c_1 - \frac{3}{2}c_2)\cos 3t$
$(\frac{3}{2}c_1 - \frac{3}{2}c_2)\sin 3t]$

$x(0) = 3 \Rightarrow -\frac{3}{2}c_1 - \frac{3}{2}c_2 = 3$

$y(0) = 2 \Rightarrow c_1 = 2$

$\Rightarrow c_1 = 2 \;\&\; c_2 = -4$

$\Rightarrow x(t) = e^{-2t}(3\cos 3t + 9\sin 3t)$

$y(t) = e^{-2t}(2\cos 3t - 4\sin 3t)$

P9. $\begin{cases} x' = 2x - 3y + 2\sin 2t & —① \\ y' = x - 2y - \cos 2t & —② \end{cases}$

From ②

$x = y' + 2y + \cos 2t$

$x' = y'' + 2y' - 2\sin 2t$

put into ①

$y'' + 2y' - 2\sin 2t = 2(y' + 2y + \cos 2t)$
$- 3y + 2\sin 2t$

$$y'' - 4y = 4\sin 2t + 2\cos 2t \rightarrow (r^2 - 4 = 0)$$

By method of undetermined coefficients to get

$$y(t) = C_1 e^{-2t} + C_2 e^{2t} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t$$

$$\Rightarrow y'(t) = -2C_1 e^{-2t} + 2C_2 e^{2t} + \frac{1}{2}\sin 2t - \cos 2t$$

$$\Rightarrow x(t) = y' + 2y + \cos 2t$$

$$= 4C_2 e^{2t} - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$$

P12
$$\begin{cases} x'' = 6x + 2y & \text{— ①} \\ y'' = 3x + 7y & \text{— ②} \end{cases}$$

From ②

$$x = \frac{1}{3}y'' - \frac{7}{3}y$$

$$x'' = \frac{1}{3}y^{(4)} - \frac{7}{3}y''$$

put into ①

$$\frac{1}{3}y^{(4)} - \frac{7}{3}y'' = 2y'' - 14y + 2y$$

$$y^{(4)} - 13y'' + 36y = 0 \quad (r^4 - 13r + 36 = 0)$$

$$\Rightarrow y(t) = (C_1 + C_2 t)e^{2t} + (C_3 + C_4 t)e^{3t}$$

$$\Rightarrow y''(t) = (C_1 + 3C_2 + 4C_2 t)e^{2t} + (C_3 + 4C_4 + 9C_4 t)e^{3t}$$

$$\Rightarrow x(t) = \frac{1}{3}y'' - \frac{7}{3}y$$

$$= (-2C_1 + C_2 - C_2 t)e^{2t} + \left(-2C_3 + \frac{4}{3}C_4 + \frac{2}{3}C_4 t\right)e^{3t}$$