§3.4.

P1. Solution:

$k = 16\ N/m,\ m = 4\ kg$

$\Rightarrow \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{4}} = 2\ rad/s$

and

$x'' + 4x = 0$

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi \approx 3.14\,s$

$\nu = \frac{1}{T} = \frac{1}{\pi} \approx 0.318\ Hz.$

P2. solution:

$k = 48\ N/m,\ m = 0.75\,kg.$

$\Rightarrow W_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{48}{0.75}} = \sqrt{64} = 8$

and

$x'' + 64x = 0$

$T = \frac{2\pi}{W_0} = \frac{2\pi}{8} = \frac{\pi}{4} \approx 0.785\,s$

$\nu = \frac{1}{T} = \frac{4}{\pi} = 1.274\ Hz.$

P3. Solution:

$k = \frac{15}{0.2} = 75\ N/m,\ m = 3\,kg.$

$\Rightarrow W_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5\ rad/s$

and

$x'' + 25x = 0.$

$T = \frac{2\pi}{w_0} = \frac{2\pi}{5} \approx 1.256\,s$

$\nu = \frac{1}{T} = \frac{5}{2\pi} \approx 0.796\ Hz$

$x(t) = A\cos 5t + B\sin 5t$

with $x'(t) = -5A\sin 5t + 5B\cos 5t$

$$\left.\begin{array}{l} x(0) = 0 \Rightarrow A = 0 \\ x'(0) = -10 \Rightarrow 5B = -10 \end{array}\right\} \Rightarrow \begin{array}{l} A = 0 \\ B = -2 \end{array}$$

$\Rightarrow x(t) = -2\sin 5t$

$\Rightarrow$ The Amplitude

$C = \sqrt{0^2 + (-2)^2} = 2\,m.$

§3.5.

P1. solution:

step 1. find the complementary solution y_c.

Characteristic equation:

$r^2 + 16 = 0$

$r = \pm 4i$

$\Rightarrow \underline{y_c(x) = C_1\cos 4x + C_2\sin 4x}$

step 2. find a particular solution y_p.

$f(x) = e^{3x}$ no duplication.

thus

$y_p(x) = Ae^{3x}$

with $y_p'(x) = 3Ae^{3x}$

$y_p''(x) = 9Ae^{3x}$

Substituting y_p into the nonhomogeneous equation.

$y_p'' + 16y_p = 9Ae^{3x} + 16(Ae^{3x})$

$= 25Ae^{3x}$

$= e^{3x}$

$\Rightarrow 25A = 1 \Rightarrow A = \frac{1}{25}$

Thus $\underline{y_p(x) = \frac{1}{25}e^{3x}}$

P2. solution:

step 1. find y_c.

Characteristic equation:

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2,\ r_2 = -1$

$\Rightarrow y_c(x) = C_1e^{2x} + C_2e^{-x}$

step 2. find y_p. no duplication.

$f(x) = 3x + 4$

Then take

$y_p(x) = Ax + B$

with $y_p'(x) = A$

$y_p''(x) = 0$

$$y_p'' - y_p' - 2y_p = 0 - A - 2(AX+B)$$
$$= -2AX - A - 2B$$
$$= 3X + 4$$

$$\Rightarrow \left.\begin{array}{l} -2A = 3 \\ -A - 2B = 4 \end{array}\right\} \Rightarrow \begin{array}{l} A = -\frac{3}{2} \\ B = -\frac{5}{4} \end{array}$$

Thus $\underline{y_p(x) = -\frac{3}{2}x - \frac{5}{4}}$

P3. Solution:

step 1. find y_c

characteristic equation.

$$r^2 - r - 6 = 0$$
$$(r-3)(r+2) = 0$$
$$r_1 = 3,\ r_2 = -2$$
$$\Rightarrow y_c(x) = \underline{C_1 e^{3x} + C_2 e^{-2x}}$$

step 2. find y_p. no duplication.

$$f(x) = \underline{2\sin 3x}$$

Then take

$$y_p(x) = A\sin 3x + B\cos 3x$$

with $y_p'(x) = 3A\cos 3x - 3B\sin 3x$

$y_p''(x) = -9A\sin 3x - 9B\cos 3x$

$$\Rightarrow y_p'' - y_p' - 6y_p = [-9A\sin 3x - 9B\cos 3x] - [3A\cos 3x - 3B\sin 3x] - 6[A\sin 3x + B\cos 3x]$$
$$= (-15A + 3B)\sin 3x + (-3A - 15B)\cos 3x$$
$$= 2\sin 3x$$

$$\Rightarrow \left.\begin{array}{l} -15A + 3B = 2 \\ -3A - 15B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A = \frac{-5}{39} \\ B = \frac{1}{39} \end{array}$$

Thus, $\underline{y_p(x) = -\frac{5}{39}\sin 3x + \frac{1}{39}\cos 3x}$

P5. Solution:

step 1. find y_c

Characteristic equation:

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\Rightarrow y_c(x) = \underline{e^{-\frac{1}{2}x}\left(C_1\cos\tfrac{\sqrt{3}}{2}x + C_2\sin\tfrac{\sqrt{3}}{2}x\right)}$$

step 2: find y_p. no duplication

$$f(x) = \sin^2 x = \frac{1-\cos 2x}{2} = \underline{\frac{1}{2}} - \underline{\frac{\cos 2x}{2}}$$

then take

$$y_p(x) = A + B\cos 2x + C\sin 2x$$

with $y_p'(x) = -2B\sin 2x + 2C\cos 2x$

$y_p''(x) = -4B\cos 2x - 4C\sin 2x$

$$\Rightarrow y_p'' + y_p' + y_p = (-4B\cos 2x - 4C\sin 2x) + (-2B\sin 2x + 2C\cos 2x) + (A + B\cos 2x + C\sin 2x)$$
$$= A + (-3B + 2C)\cos 2x + (-3C - 2B)\sin 2x$$
$$= \frac{1}{2} - \frac{\cos 2x}{2}$$

$$\Rightarrow \left.\begin{array}{l} A = \frac{1}{2} \\ -3B + 2C = -\frac{1}{2} \\ -3C - 2B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A = \frac{1}{2} \\ B = \frac{3}{26} \\ C = -\frac{1}{13} \end{array}$$

Thus,

$$\underline{y_p(x) = \frac{1}{2} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x}$$

P. Solution:

step 1: find y_c

Characteristic equation:

$$r^2 - 4 = 0$$
$$r_1 = r_2 = 2 \text{ (double root)}$$
$$\Rightarrow y_c(x) = \underline{(C_1 + C_2 x)e^{2x}}$$

no duplication.

step 2: find y_p

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2} = \underline{\frac{1}{2}e^x} - \underline{\frac{1}{2}e^{-x}}$$

Then take.

$$y_p(x) = Ae^x + Be^{-x}$$

with $y_p'(x) = Ae^x - Be^{-x}$

$y_p''(x) = Ae^x + Be^{-x}$

$$\Rightarrow y_p'' - 4y_p = (Ae^x + Be^{-x}) - 4(Ae^x + Be^{-x})$$
$$= -3Ae^x - 3Be^{-x}$$
$$= \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}$$
$$\Rightarrow \left.\begin{aligned} -3A &= \tfrac{1}{2} \\ -3B &= -\tfrac{1}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\tfrac{1}{6} \\ B &= \tfrac{1}{6} \end{aligned}$$

Thus,
$$y_p(x) = -\tfrac{1}{6}e^x + \tfrac{1}{6}e^{-x}$$
$$= -\tfrac{1}{6}(e^x - e^{-x})$$
$$= \underline{-\tfrac{1}{3}\sinh x}$$

P9. Solution:

step 1. find y_c.

Characteristic equation:
$$r^2 + 2r - 3 = 0$$
$$(r+3)(r-1) = 0$$
$$r_1 = -3,\ r_2 = 1$$
$$\Rightarrow y_c(x) = c_1 e^{-3x} + \underline{c_2 e^x}$$

step 2. find y_p
$$f(x) = \underline{1} + \underline{xe^x}$$
take
$$y_p = A + \underbrace{x^{(s)}(B + Cx)e^x}_{s=1 \text{ to avoid duplication.}}$$

Thus
$$y_p = A + (Bx + Cx^2)e^x$$
with
$$y_p' = [B + (B+2C)x + Cx^2]e^x$$
$$y_p'' = [(2B+2C) + (B+4C)x + Cx^2]e^x$$
$$\Rightarrow y_p'' + 2y_p' - 3y_p = -3A + [(4B+2C) + 8Cx]e^x$$
$$= 1 + xe^x$$
$$\Rightarrow \left.\begin{aligned} -3A &= 1 \\ 4B + 2C &= 0 \\ 8C &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \tfrac{1}{3} \\ B &= -\tfrac{1}{16} \\ C &= \tfrac{1}{8} \end{aligned} \Rightarrow \underline{y_p(x) = \tfrac{1}{3} + (\tfrac{1}{8}x^2 - \tfrac{1}{16}x)e^x}$$

P11. Solution:

Step 1. find $y_c(x)$

Characteristic equation:
$$r^3 + 4r = 0$$
$$r(r^2 + 4) = 0$$
$$r_1 = 0,\ r_{2,3} = \pm 2i$$
$$\Rightarrow y_c(x) = \underline{C_1} + C_2\cos 2x + C_3 \sin 2x$$

step 2. find $y_p(x)$
$$f(x) = \underline{3x - 1}$$
set
$$y_p(x) = \underbrace{x^{(s)}(A + Bx)}_{s=1 \text{ to avoid duplication}}$$

then take
$$y_p(x) = Ax + Bx^2$$
with
$$y_p' = A + 2Bx$$
$$y_p'' = 2B$$
$$y_p^{(3)} = 0$$
$$\Rightarrow y_p^{(3)} + 4y_p' = 0 + 4(A + 2Bx)$$
$$= 4A + 8Bx$$
$$= -1 + 3x$$
$$\Rightarrow \left.\begin{aligned} 4A &= -1 \\ 8B &= 3 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\tfrac{1}{4} \\ B &= \tfrac{3}{8} \end{aligned}$$

Thus, $\underline{y_p(x) = -\tfrac{1}{4}x + \tfrac{3}{8}x^2}$

P12. Solution:

Step 1. find $y_c(x)$

Characteristic equation:
$$r^3 + r = 0$$
$$r(r^2 + 1) = 0$$
$$r_1 = 0,\ r_{2,3} = \pm i$$
$$\Rightarrow y_c(x) = \underline{C_1} + \underline{C_2\cos x + C_3 \sin x}$$

step 2. find $y_p(x)$
$$f(x) = \underline{2} - \underline{\sin x}$$

to avoid duplication

set
$$y_p(x) = \underbrace{x^{(s_1)}A}_{s_1 = 1} + \underbrace{x^{(s_2)}(B\cos x + C\sin x)}_{s_2 = 1}$$

Thus, we take

$y_p(x) = Ax + Bx\cos x + Cx\sin x$.

with

$y_p'(x) = A + B\cos x - Bx\sin x + C\sin x + Cx\cos x$

$y_p''(x) = -2B\sin x - Bx\cos x + 2C\cos x - Cx\sin x$

$y_p^{(3)}(x) = -3B\cos x + Bx\sin x - 3C\sin x - Cx\cos x$

$$\Rightarrow y_p^{(3)} + y_p' = (-3B\cos x + Bx\sin x - 3C\sin x - Cx\cos x) + (Ax + B\cos x - Bx\sin x + C\sin x + Cx\cos x)$$

$$= A - 2B\cos x - 2C\sin x$$

$$= 2 - \sin x$$

$$\Rightarrow \left.\begin{aligned} A &= 2 \\ -2B &= 0 \\ -2C &= -1 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 2 \\ B &= 0 \\ C &= \tfrac{1}{2} \end{aligned}$$

Thus, $\underline{y_p(x) = 2x + \frac{1}{2}x\sin x}$

P13. Solution:

step 1. find y_c.

$r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{2^2 - 4\cdot1\cdot5}}{2} = -1 \pm 2i$

$\Rightarrow y_c(x) = \underline{e^{-x}(C_1\cos 2x + C_2\sin 2x)}$

step 2. find y_p

$f(x) = \underline{e^x\sin x}$

no duplication

then we take

$y_p(x) = e^x(A\cos x + B\sin x)$

with $y_p'(x) = e^x[(A+B)\cos x + (B-A)\sin x]$

$y_p''(x) = e^x[2B\cos x - 2A\sin x]$

$$\Rightarrow y_p'' + 2y_p' + 5y_p = e^x[2B\cos x - 2A\sin x] + 2e^x[(A+B)\cos x + (B-A)\sin x] + 5e^x[A\cos x + B\sin x]$$

$$= e^x[(7A+4B)\cos x + (-4A+7B)\sin x]$$

$$= e^x\sin x$$

$$\Rightarrow \left.\begin{aligned} 7A + 4B &= 0 \\ -4A + 7B &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\tfrac{4}{65} \\ B &= \tfrac{7}{65} \end{aligned}$$

Thus, $\underline{y_p(x) = e^x\left[-\frac{4}{65}\cos x + \frac{7}{65}\sin x\right]}$

P17. Solution:

step 1. find y_c.

characteristic equation:

$r^2 + 1 = 0$

$r_{1,2} = \pm i$

$\Rightarrow y_c(x) = \underline{C_1\cos x + C_2\sin x}$.

step 2. find y_p.

$f(x) = \sin x + x\cos x$

we form

$y_p(x) = x^{s}[(A+Bx)\cos x + (C+Dx)\sin x]$

s=1 to avoid duplication

thus take

$y_p(x) = (Ax + Bx^2)\cos x + (Cx + Dx^2)\sin x$.

with $y_p' = [A + (2B+C)x + Dx^2]\cos x + [C + (2D-A)x - Bx^2]\sin x$

$y_p'' = [(2B+2C) + (4D-A)x - Bx^2]\cos x + [(2D-2A) + (-4B-C)x - Dx^2]\sin x$

$$\Rightarrow y_p'' + y_p = [(2B+2C) + 4Dx]\cos x + [(2D-2A) - 4Bx]\sin x$$

$$= \sin x + x\cos x$$

$$\Rightarrow \left.\begin{aligned} 2B + 2C &= 0 \\ 4D &= 1 \\ 2D - 2A &= 1 \\ -4B &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\tfrac{1}{4} \\ B &= 0 \\ C &= 0 \\ D &= \tfrac{1}{4} \end{aligned}$$

Thus,

$\underline{y_p(x) = -\frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x}$