§3.2

P1. Solution:

$$\left(\frac{5}{2}\right)(2x)+\left(\frac{8}{3}\right)(3x^2)+(-1)(5x-8x^2)=0$$

$$\frac{5}{2}f(x)-\frac{8}{3}g(x)-h(x)=0$$

P4. Solution:

$$-\frac{3}{17}f(x)-\frac{3}{2}g(x)+h(x)=0$$

P7. Solution:

$$W(f,g,h)=\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$=2\neq 0$$

P8. Solution:

$$W(f,g,h)=\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$=2e^{6x}\neq 0$$

P10. Solution:

$$W(f,g,h)=\begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & -2x^{-3}\ln x+x^{-3} \\ e^x & 6x^{-4} & 6x^{-4}\ln x-5x^{-4} \end{vmatrix}$$

$$=e^x\left(4x^{-7}+5x^{-6}+x^{-5}\right)$$

$\neq 0$ when $x>0$.

P13. Solution:

$$y(x)=c_1\underbrace{e^x}_{y_1}+c_2\underbrace{e^{-x}}_{y_2}+c_3\underbrace{e^{-2x}}_{y_3}$$

$$y'(x)=c_1e^x-c_2e^{-x}-2c_3e^{-2x}$$

$$y''(x)=c_1e^x+c_2e^{-x}+4c_3e^{-2x}$$

$$\Rightarrow \left.\begin{aligned} y(0)&=c_1+c_2+c_3=1 \\ y'(0)&=c_1-c_2-2c_3=2 \\ y''(0)&=c_1+c_2+4c_3=0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1&=\tfrac{4}{3} \\ c_2&=0, \\ c_3&=-\tfrac{1}{3} \end{aligned}$$

Thus, $y(x)=\frac{4}{3}e^x-\frac{1}{3}e^{-2x}$

P17. Solution:

$$y(x)=c_1+c_2\cos 3x+c_3\sin 3x$$

$$y'(x)=-3c_2\sin 3x+3c_3\cos 3x$$

$$y''(x)=-9c_2\cos 3x-9c_3\sin 3x$$

$$\Rightarrow \begin{aligned} y(0)&=c_1+c_2=3 \\ y'(0)&=3c_3=-1 \\ y''(0)&=-9c_2=2. \end{aligned} \Rightarrow \begin{aligned} c_1&=\tfrac{29}{9} \\ c_2&=-\tfrac{2}{9} \\ c_3&=-\tfrac{1}{3} \end{aligned}$$

Thus, $y(x)=\frac{29}{9}-\frac{2}{9}\cos 3x-\frac{1}{3}\sin 3x$

P19. Solution:

$$y(x)=c_1x+c_2x^2+c_3x^3$$

$$y'(x)=c_1+2c_2x+3c_3x^2$$

$$y''(x)=2c_2+6c_3x$$

$$\Rightarrow \left.\begin{aligned} y(1)&=c_1+c_2+c_3=6 \\ y'(1)&=c_1+2c_2+3c_3=14 \\ y''(1)&=2c_2+6c_3=22 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1&=1 \\ c_2&=2 \\ c_3&=3 \end{aligned}$$

Thus $y(x)=x+2x^2+3x^3$

P21. Solution:

$$y(x)=y_c(x)+y_p(x)$$

$$=c_1\cos x+c_2\sin x+3x$$

$$y'(x)=-c_1\sin x+c_2\cos x+3$$

$$\Rightarrow \left.\begin{aligned} y(0)&=c_1=2 \\ y'(0)&=c_2+3=-2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1&=2 \\ c_2&=-5 \end{aligned}$$

Thus, $y(x)=2\cos x-5\sin x+3x$

P24. Solution:

$y(x) = y_c + y_p = C_1 e^x \cos x + C_2 e^x \sin x + x + 1$

$y'(x) = C_1 e^x \cos x - C_1 e^x \sin x + C_2 e^x \sin x + C_2 e^x \cos x$
$+ 1$

$= C_1 e^x (\cos x - \sin x) + C_2 e^x (\sin x + \cos x) + 1$

$\Rightarrow \left.\begin{aligned} y(0) &= C_1 + 1 = 4 \\ y'(0) &= C_1 + C_2 + 1 = 8 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 3 \\ C_2 &= 4 \end{aligned}$

Thus, $y(x) = 3e^x \cos x + 4e^x \sin x + x + 1$.

§3.3.

P1. Solution: Characteristic equation:

$r^2 - 4 = 0$

$\Rightarrow r_1 = 2,\ r_2 = -2$

$\Rightarrow y_1(x) = e^{2x},\ y_2(x) = e^{-2x}$

Thus, $y(x) = C_1 e^{2x} + C_2 e^{-2x}$

P2. Solution: Characteristic equation:

$2r^2 - 3r = 0$

$r(2r - 3) = 0$

$r_1 = 0,\ r_2 = \frac{3}{2}$

$\Rightarrow y_1(x) = 1,\ y_2 = e^{\frac{3}{2}x}$

Thus, $y(x) = C_1 + C_2 e^{\frac{3}{2}x}$

P4. Solution: Characteristic equation:

$2r^2 - 7r + 3 = 0$

$(2r - 1)(r - 3) = 0$

$r_1 = \frac{1}{2},\ r_2 = 3$

$\Rightarrow y_1(x) = e^{\frac{1}{2}x},\ y_2(x) = e^{3x}$

Thus, $y(x) = C_1 e^{\frac{1}{2}x} + C_2 e^{3x}$

P8. Solution: Characteristic equation:

$r^2 - 6r + 13 = 0$

$r = 3 \pm 2i$

Thus, $y(x) = e^{3x}(C_1 \cos 2x + C_2 \sin 2x)$

P11. Solution: Characteristic equation:

$r^4 - 8r^3 + 16r^2 = 0$

$r^2(r^2 - 8r + 16) = 0$

$r^2(r - 4)^2 = 0$

$\Rightarrow \begin{cases} r_1 = 0 & \text{(double root } k = 2) \\ r_2 = 4 & \text{(double root } k = 2) \end{cases}$

$r_1 = 0$ Contributes $(C_1 + C_2 x)e^{0x}$
$= (C_1 + C_2 x)$

$r_2 = 4$ Contributes $(C_3 + C_4 x)e^{4x}$

Thus, $y(x) = C_1 + C_2 x + (C_3 + C_4 x)e^{4x}$

P14. Solution: Characteristic equation

$r^4 + 3r^2 - 4 = 0$

$(r^2 + 4)(r^2 - 1) = 0$

$\Rightarrow r^2 = -4$ or $r^2 = 1$

$\Rightarrow r_{1,2} = \pm 2i,\ r_3 = 1,\ r_4 = -1$.

$r_{1,2} = \pm 2i$ Contribute $C_1 \cos 2x + C_2 \sin 2x$

$r_3 = 1$ Contributes $C_3 e^x$

$r_4 = -1$ Contributes $C_4 e^{-x}$

Thus, $y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 e^x + C_4 e^{-x}$

P16. Solution: Characteristic equation.

$r^4 - 8r^2 + 16 = 0$

$(r^2 - 4)^2 = 0$

$(r - 2)^2 (r + 2)^2 = 0$

$r_1 = 2$ (double root)

$r_2 = -2$ (double root)

$r_1 = 2$ Contributes $(C_1 + C_2 x)e^{2x}$

$r_2 = -2$ Contributes $(C_3 + C_4 x)e^{-2x}$

Thus, $y(x) = (C_1 + C_2 x)e^{2x} + (C_3 + C_4 x)e^{-2x}$

P21. Solution: Characteristic equation:

$$r^2 - 4r + 3 = 0$$

$$\Rightarrow (r-3)(r-1) = 0$$

$$\Rightarrow r_1 = 3, \quad r_2 = 1$$

$$\Rightarrow y(x) = c_1 e^{3x} + c_2 e^{x}$$

$$y'(x) = 3c_1 e^{3x} + c_2 e^{x}$$

$$\left.\begin{aligned} y(0) &= C_1 + C_2 = 7 \\ y'(0) &= 3C_1 + C_2 = 11 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 5 \end{aligned}$$

Thus, $y(x) = 2e^{3x} + 5e^{x}$

P24. Solution: Characteristic equation:

$$2r^3 - 3r^2 - 2r = 0$$

$$r(2r^2 - 3r - 2) = 0$$

$$r(2r+1)(r-2) = 0$$

$$\Rightarrow r_1 = 0, \; r_2 = -\tfrac{1}{2}, \; r_3 = 2$$

Thus, $y(x) = C_1 + C_2 e^{-\frac{1}{2}x} + C_3 e^{2x}$

$$y'(x) = -\tfrac{1}{2}C_2 e^{-\frac{1}{2}x} + 2C_3 e^{2x}$$

$$y''(x) = \tfrac{1}{4}C_2 e^{-\frac{1}{2}x} + 4C_3 e^{2x}$$

$$\left.\begin{aligned} y(0) &= C_1 + C_2 + C_3 = 1 \\ y'(0) &= -\tfrac{1}{2}C_2 + 2C_3 = -1 \\ y''(0) &= \tfrac{1}{4}C_2 + 4C_3 = 3 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= -\tfrac{7}{2} \\ C_2 &= 4 \\ C_3 &= \tfrac{1}{2} \end{aligned}$$

Thus, $y(x) = -\frac{7}{2} + 4e^{-\frac{1}{2}x} + \frac{1}{2}e^{2x}$

P26. Solution: Characteristic equation:

$$r^3 + 10r^2 + 25r = 0$$

$$r(r+5)^2 = 0$$

$$r = 0, \quad r = -5 \text{ (double root)}$$

Thus $y(x) = C_1 + (C_2 + C_3 x)e^{-5x}$

$$y'(x) = -5C_2 e^{-5x} + C_3[1 - 5x]e^{-5x}$$

$$y''(x) = 25C_2 e^{-5x} + C_3[-10 + 25x]e^{-5x}$$

$$\left.\begin{aligned} \Rightarrow y(0) &= C_1 + C_2 = 3 \\ y'(0) &= -5C_2 + C_3 = 4 \\ y''(0) &= 25C_2 - 10C_3 = 5 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= \tfrac{24}{5} \\ C_2 &= -\tfrac{9}{5} \\ C_3 &= -5 \end{aligned}$$

$$\Rightarrow y(x) = \frac{24}{5} + \left(-\frac{9}{5} - 5x\right)e^{-5x}$$

P33. Solution: Characteristic equation:

$$r^3 + 3r^2 - 54 = 0$$

$y = e^{3x}$ is a solution $\Rightarrow r = 3$ is a root of the characteristic equation.

$$(r-3)(r^2 + 6r + 18) = 0$$

$$\Rightarrow r_1 = 3, \quad r_{2,3} = -3 \pm 3i$$

$r_{2,3} = -3 \pm 3i$ Contribute $e^{-3x}(C_2 \cos 3x + C_3 \sin 3x)$

Thus, $y(x) = C_1 e^{3x} + e^{-3x}(C_2 \cos 3x + C_3 \sin 3x)$