§3.1.

P1. Solution:

$y_1 = e^x,\ y_1' = e^x,\ y_1'' = e^x$

$\Rightarrow y_1'' - y_1 = 0$

$y_2 = e^{-x},\ y_2' = -e^{-x},\ y_2'' = e^{-x}$

$\Rightarrow y_2'' - y_2 = 0$

$$\left.\begin{aligned} y(x) &= c_1e^x + c_2e^{-x} \\ y'(x) &= c_1e^x - c_2e^{-x}\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= c_1 - c_2 = 5\end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 5/2 \\ c_2 &= -5/2\end{aligned}$$

Thus, $y(x) = \frac{5}{2}e^x - \frac{5}{2}e^{-x}$

P4 Solution:

$y_1 = \cos 5x,\ y_1' = -5\sin 5x,\ y_1'' = -25\cos 5x$

$y_2 = \sin 5x,\ y_2' = 5\cos 5x,\ y_2'' = -25\sin 5x$

$\Rightarrow y_1'' + 25y_1 = 0,\ y_2'' + 25y_2 = 0$

$$\left.\begin{aligned} y(x) &= c_1\cos 5x + c_2\sin 5x \\ y'(x) &= -5c_1\sin 5x + 5c_2\cos 5x\end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} y(0) &= c_1 = 10 \\ y'(0) &= 5c_2 = -10\end{aligned} \Rightarrow \begin{aligned} c_1 &= 10 \\ c_2 &= -2\end{aligned}$$

Thus, $y(x) = 10\cos 5x - 2\sin 5x$

P6. Solution:

$y_1 = e^{2x},\ y_1' = 2e^{2x},\ y_1'' = 4e^{2x}$

$\Rightarrow y_1'' + y_1' - 6y_1 = 0$

$y_2 = e^{-3x},\ y_2' = -3e^{-3x},\ y_2 = 9e^{-3x}$

$\Rightarrow y_2'' + y_2' - 6y_2 = 0$

$$\left.\begin{aligned} y(x) &= c_1e^{2x} + c_2y^{-3x} \\ y'(x) &= 2c_1e^{2x} - 3c_2y^{-3x}\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 7 \\ y'(0) &= 2c_1 - 3c_2 = -1\end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 4 \\ c_2 &= 3\end{aligned}$$

Thus, $y(x) = 4e^{2x} + 3e^{-3x}$

P9. Solution:

$y_1 = e^{-x},\ y_1' = -e^{-x},\ y_1'' = e^{-x}$

$\Rightarrow y_1'' + 2y_1' + y_1 = 0$

$y_2 = xe^{-x},\ y_2' = -xe^{-x} + e^{-x},\ y_2'' = xe^{x} - e^{-x} - e^{-x}$
$= xe^{x} - 2e^{-x}$

$\Rightarrow y_2'' + 2y_2' + y_2 = xe^{x} - 2e^{-x} + 2(-xe^{-x} + e^{-x}) + xe^{-x} = 0$

$y(x) = c_1e^{-x} + c_2xe^{-x}$

$y'(x) = -c_1e^{-x} + c_2(e^{-x} - xe^{-x})$

$$\Rightarrow \begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= -c_1 + c_2 = -1\end{aligned} \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 1\end{aligned}$$

Thus, $y(x) = 2e^{-x} + xe^{-x}$

P12. Solution:

$y_1 = e^{-3x}\cos 2x,\ y_1' = -3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x$

$y_1'' = 9e^{-3x}\cos 2x + 6e^{-3x}\sin 2x + 6e^{-3x}\sin 2x - 4e^{-3x}\cos 2x$

$= 5e^{-3x}\cos 2x + 12e^{-3x}\sin 2x$

$y_2 = e^{-3x}\sin 2x,\ y_2' = -3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x$

$y_2'' = 9e^{-3x}\sin 2x - 6e^{-3x}\cos 2x - 6e^{-3x}\cos 2x - 4e^{-3x}\sin 2x$

$= -12e^{-3x}\cos 2x + 5e^{-3x}\sin 2x$

$\Rightarrow y_1'' + 6y_1' + 13y_1 = 5e^{-3x}\cos 2x + 12e^{-3x}\sin 2x + 6(-3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x) + 13e^{-3x}\cos 2x$

$= 0$

$y_2'' + 6y_2' + 13y_2 = -12e^{-3x}\cos 2x + 5e^{-3x}\sin 2x + 6(-3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x) + 13e^{-3x}\sin 2x$

$= 0$

$y(x) = c_1e^{-3x}\cos 2x + c_2e^{-3x}\sin 2x$

$y'(x) = c_1(-3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x) + c_2(-3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x)$

$$\Rightarrow \begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= c_1(-3) + c_2(2) = 0\end{aligned} \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 3\end{aligned}$$

Thus $y(x) = 2e^{-3x}\cos 2x + 3e^{-3x}\sin 2x$

P20. Solution:

$$W(f,g)=\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} \pi & \underbrace{\cos^2 x+\sin^2 x}_{=1} \\ 0 & 0 \end{vmatrix}$$

$$= 0$$

$\Rightarrow$ linearly dependent

P22. Solution:

$f(x)=1+x,\quad g(x)=1+|x|$

$f'(x)=1,\quad g'(x)=\begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$ not differentiable at $x=0$.

clearly $1+x \not\equiv c(1+|x|)\ \forall x\in\mathbb{R}$ for any c

and $c(1+x)\not\equiv 1+|x|\ \forall x\in\mathbb{R}$ for any c.

$\Rightarrow$ linearly independent.

P25. Solution:

$f(x)=e^x\sin x,\quad g(x)=e^x\cos x$

$f'(x)=e^x\sin x+e^x\cos x$

$g'(x)=e^x\cos x-e^x\sin x$

$$W(f,g)=\begin{vmatrix} e^x\sin x & e^x\cos x \\ e^x(\sin x+\cos x) & e^x(\cos x-\sin x) \end{vmatrix}$$

$$= e^{2x}(\sin x\cos x-\sin^2 x)$$

$$-e^{2x}(\sin x\cos x+\cos^2 x)$$

$$= -e^{2x}\neq 0$$

$\Rightarrow$ linearly independent

P32. Solution:

Characteristic equation: $r^2-3r+2=0$

$\Rightarrow (r-2)(r-1)=0$

$r_1=2,\ r_2=1$

Thus $y_1(x)=e^{2x},\ y_2=e^x$

$\Rightarrow y(x)=c_1e^{2x}+c_2e^x$

P38. Solution:

Characteristic equation: $4r^2+8r+3=0$

$\Rightarrow (2r+1)(2r+3)=0$

$\Rightarrow r_1=-\frac{1}{2},\ r_2=-\frac{3}{2}$

Thus, $y_1=e^{-\frac{1}{2}x},\ y_2=e^{-\frac{3}{2}x}$

$\Rightarrow y(x)=c_1e^{-\frac{1}{2}x}+c_2e^{-\frac{3}{2}x}$

P39. Solution:

Characteristic equation: $4r^2+4r+1=0$

$\Rightarrow (2r+1)^2=0 \Rightarrow r_1=r_2=-\frac{1}{2}$

Thus, $y_1(x)=e^{-\frac{1}{2}x},\ y_2(x)=xe^{-\frac{1}{2}x}$

$\Rightarrow y(x)=c_1e^{-\frac{1}{2}x}+c_2xe^{-\frac{1}{2}x}$

P40. Solution:

Characteristic equation: $9r^2-12r+4=0$

$\Rightarrow (3r-2)^2=0 \Rightarrow r=\frac{2}{3}$

Thus, $y_1(x)=e^{\frac{2}{3}x},\ y_2(x)=xe^{\frac{2}{3}x}$

$\Rightarrow y(x)=c_1e^{\frac{2}{3}x}+c_2xe^{\frac{2}{3}x}$