

§2.2.

P1. solution:  $\frac{dx}{dt} = x - 4$

$x - 4 = 0 \Rightarrow x = 4 \rightarrow$  critical point



$\int \frac{1}{x-4} dx = \int dt$

$\ln|x-4| = t + C$

$x - 4 = Ae^t$

$x(0) = X_0 \Rightarrow X_0 - 4 = A \Rightarrow$

$x - 4 = (X_0 - 4)e^t$

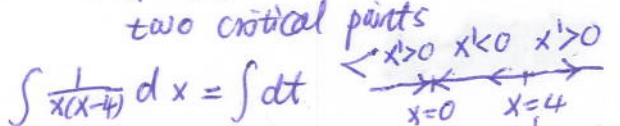
$x(t) = 4 + (X_0 - 4)e^t$

P3. solution:  $\frac{dx}{dt} = x^2 - 4x$

$x^2 - 4x = 0 \Rightarrow x(x-4) = 0$

$\Rightarrow x = 0$  or  $x = 4$

two critical points



$\int \frac{1}{x(x-4)} dx = \int dt$

$\int \left( \frac{-1/4}{x} - \frac{1/4}{x-4} \right) dx = \int dt$

$\ln|x| - \ln|x-4| = -4t + C$

$\frac{x}{x-4} = Ae^{-4t}$

$x(0) = X_0 \Rightarrow \frac{X_0}{X_0-4} = A$

$\Rightarrow \frac{x}{x-4} = \frac{X_0 e^{-4t}}{X_0-4}$

$x(X_0-4) = X_0(x-4)e^{-4t}$

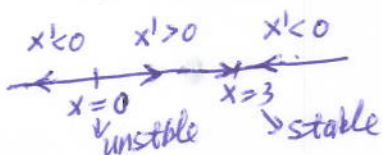
$(X_0 e^{-4t} - X_0 + 4)x = 4X_0 e^{-4t}$

$x(t) = \frac{4X_0 e^{-4t}}{X_0 e^{-4t} - X_0 + 4}$

P4. solution:  $\frac{dx}{dt} = 3x - x^2$

$3x - x^2 = 0 \Rightarrow x(3-x) = 0$

$\Rightarrow x = 0$  or  $x = 3$



$\int \frac{1}{x(3-x)} dx = \int dt$

$\int \frac{1}{3} \left( \frac{1}{x} + \frac{1}{3-x} \right) dx = t + C$

$\ln|x| - \ln|3-x| = 3t + C$

$\ln \left| \frac{x}{3-x} \right| = 3t + C$

$\frac{x}{3-x} = Ae^{3t}$

$x(0) = X_0 \Rightarrow A = \frac{X_0}{3-X_0}$

$\Rightarrow \frac{x}{3-x} = \frac{X_0 e^{3t}}{3-X_0}$

$x(3-X_0) = X_0(3-x)e^{3t}$

$x(3-X_0 + X_0 e^{3t}) = 3X_0 e^{3t}$

$x(t) = \frac{3X_0 e^{3t}}{3-X_0 + X_0 e^{3t}}$

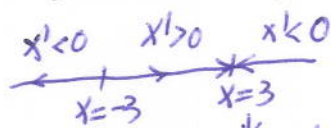
$x(t) = \frac{3X_0}{X_0 + (3-X_0)e^{-3t}}$

P6. solution:  $\frac{dx}{dt} = 9 - x^2$

$9 - x^2 = 0 \Rightarrow (3-x)(3+x) = 0$

$\Rightarrow x = 3$  or  $x = -3$

two critical points



$\int \frac{1}{(3-x)(3+x)} dx = \int dt$

$\int \frac{1}{6} \left( \frac{1}{3-x} + \frac{1}{3+x} \right) dx = t + C$

$-\ln|3-x| + \ln|3+x| = 6t + C$

$\ln \left| \frac{3+x}{3-x} \right| = 6t + C$

$\frac{3+x}{3-x} = Ae^{6t}$

$x(0) = X_0 \Rightarrow A = \frac{3+X_0}{3-X_0}$

$\Rightarrow \frac{3+x}{3-x} = \frac{3+X_0}{3-X_0} e^{6t}$

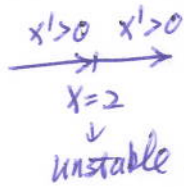
$3(3-x_0) + (3-x_0)x = 3(3+x_0)e^{6t} - x(3+x_0)e^{6t}$

$[(3-x_0) + (3+x_0)e^{6t}]x = 3(3+x_0)e^{6t} - 3(3-x_0)$

$x(t) = \frac{3(3+x_0)e^{6t} - 3(3-x_0)}{(3+x_0)e^{6t} + (3-x_0)}$

P7. Solution:  $\frac{dx}{dt} = (x-2)^2$

$(x-2)^2 = 0 \Rightarrow x=2$  critical point



$\int \frac{1}{(x-2)^2} dx = \int dt$

$-\frac{1}{x-2} = t + C$

$x(0) = x_0 \Rightarrow -\frac{1}{x_0-2} = C$

$\Rightarrow -\frac{1}{x-2} = t - \frac{1}{x_0-2}$

$-1 = t(x-2) - \frac{x-2}{x_0-2}$

$(t - \frac{1}{x_0-2})x = 2t - 1 - \frac{2}{x_0-2}$

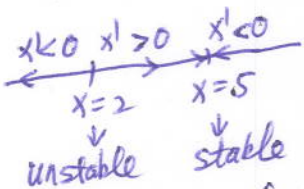
$x = \frac{(2t-1)(x_0-2) - 2}{t(x_0-2) - 1}$

$\Rightarrow x(t) = \frac{2tx_0 - 4t - x_0}{tx_0 - 2t - 1}$

P10. Solution:  $\frac{dx}{dt} = 7x - x^2 - 10$

$7x - x^2 - 10 = 0 \Rightarrow (-x+5)(x-2) = 0$

$\Rightarrow x=2$  or  $x=5$



$\int \frac{1}{(-x+5)(x-2)} dx = \int dt$

$\int \frac{1}{3} (\frac{1}{-x+5} + \frac{1}{x-2}) = t + C$

$-\ln|-x+5| + \ln|x-2| = 3t + C$

$\ln|\frac{x-2}{-x+5}| = 3t + C$

$\frac{x-2}{-x+5} = Ae^{3t}$

$x(0) = x_0 \Rightarrow A = \frac{x_0-2}{-x_0+5}$

$\Rightarrow \frac{x-2}{-x+5} = \frac{x_0-2}{-x_0+5} e^{3t}$

$(x-2)(-x+5) = (-x+5)(x_0-2)e^{3t}$   
 $[-x+5+(x_0-2)]e^{3t} = 2(-x+5) + 5(x_0-2)e^{3t}$   
 $x(t) = \frac{2(-x_0+5) + 5(x_0-2)e^{3t}}{-x_0+5 + (x_0-2)e^{3t}}$

§ 2.3

P1. Solution:

$\frac{dv}{dt} = k(250-v)$

$\int \frac{1}{250-v} dv = \int k dt$

$-\ln|250-v| = kt + C$

$\ln|250-v| = -kt + C$

$250-v = Ae^{-kt}$

$v(t) = 250 - Ae^{-kt}$

$v(0) = 0 \Rightarrow 0 = 250 - A \Rightarrow A = 250$

$\Rightarrow v(t) = 250 - 250e^{-kt}$

$v(10) = 100 \Rightarrow 250 - 250e^{-10k} = 100$

$\Rightarrow 250e^{-10k} = 150$

$\Rightarrow k = -\frac{1}{10} \ln(\frac{150}{250})$

$\approx -0.0511 \Rightarrow v(t) = 250 - 250e^{-0.0511t}$

To solve  $v(t) = 200$ , we get

$250 - 250e^{-0.0511t} = 200$

$250e^{-0.0511t} = 50$

$t = (\frac{1}{-0.0511}) \ln(\frac{50}{250})$

$\approx 31.55$

P2. Solution:

$\frac{dv}{dt} = -kv$

(a)  $\int \frac{1}{v} dv = \int -k dt$

$\ln|v| = -kt + C$

$v(t) = Ae^{-kt}$

$v(0) = v_0 \Rightarrow A = v_0 \Rightarrow v(t) = v_0 e^{-kt}$

$x(t) = x_0 + \int_0^t v(s) ds$

$= x_0 + \int_0^t v_0 e^{-ks} ds$

$= x_0 + (-\frac{v_0}{k}) e^{-ks} \Big|_{s=0}^{s=t} = x_0 + (-\frac{v_0}{k})(e^{-kt} - 1)$

$= x_0 + (\frac{v_0}{k})(1 - e^{-kt})$



$$\begin{aligned} \text{(b)} \lim_{t \rightarrow +\infty} x(t) &= \lim_{t \rightarrow +\infty} \left[ x_0 + \left( \frac{v_0}{k} \right) (1 - e^{-kt}) \right] \\ &= x_0 + \left( \frac{v_0}{k} \right) \cdot 1 \\ &= x_0 + \frac{v_0}{k} \end{aligned}$$

P3. Solution:

$$v(t) = v_0 e^{-kt} \quad \left\{ \Rightarrow v(t) = 40 e^{-kt} \right.$$

$$v_0 = 40$$

$$\begin{aligned} v(10) = 20 &\Rightarrow 40 e^{-10k} = 20 \\ &\Rightarrow k = \left( \frac{-1}{10} \right) \ln \left( \frac{20}{40} \right) \\ &\approx 0.0693 \end{aligned}$$

$$\begin{aligned} \Rightarrow \lim_{t \rightarrow +\infty} x(t) &= x_0 + \frac{v_0}{k} \\ &= \frac{40}{0.0693} \\ &\approx 577 \text{ ft.} \end{aligned}$$

P4. Solution:

$$\frac{dv}{dt} = 10 - 0.1v$$

$$\int \frac{1}{10 - 0.1v} dv = \int dt$$

$$-10 \ln |10 - 0.1v| = t + C$$

$$\ln |10 - 0.1v| = -\frac{t}{10} + C$$

$$10 - 0.1v = A e^{-t/10}$$

$$v(0) = 0 \Rightarrow 10 = A \Rightarrow$$

$$10 - 0.1v = 10 e^{-t/10}$$

$$0.1v = 10 - 10 e^{-t/10}$$

$$v = 100 - 100 e^{-t/10}$$

$$\begin{aligned} \text{(a)} \lim_{t \rightarrow +\infty} v(t) &= \lim_{t \rightarrow +\infty} [100 - 100 e^{-t/10}] \\ &= 100 \text{ ft/s} \end{aligned}$$

(b) Solve  $v(t) = 0.9 \cdot 100$ , we get

$$100 - 100 e^{-t/10} = 90$$

$$-100 e^{-t/10} = -10$$

$$t = (-10) \ln \left( \frac{10}{100} \right)$$

$$\approx 23 \text{ s}$$

$$\begin{aligned} x(t) &= x(0) + \int_0^t v(s) ds \\ &= 0 + \int_0^t (100 - 100 e^{-s/10}) ds \\ &= (100s + 1000 e^{-s/10}) \Big|_{s=0}^{s=t} \\ &= 100t + 1000(e^{-t/10} - 1) \end{aligned}$$

$$\begin{aligned} x(23) &= 23 \cdot 100 + 1000(e^{-23/10} - 1) \\ &= 2300 + 1000(-0.8997) \\ &\approx 1400.26 \text{ ft} \end{aligned}$$

P8. Solution:

$$\frac{dv}{dt} = 10 - 0.001v^2$$

$$\int \frac{1}{10 - 0.001v^2} dv = \int dt$$

$$v(t) = \sqrt{\frac{10}{0.001}} \tanh(C - \sqrt{10(0.001)} t)$$

$$= 100 \tanh(0.1t - C)$$

$$\begin{aligned} v(0) = 0 &\Rightarrow 0 = 100 \tanh(-C) \\ &\Rightarrow C = 0 \end{aligned}$$

$$\Rightarrow v(t) = 100 \tanh(0.1t)$$

$$\begin{aligned} \text{(a)} \lim_{t \rightarrow +\infty} v(t) &= \lim_{t \rightarrow +\infty} 100 \tanh(0.1t) \\ &= 100 \end{aligned}$$

$$\text{(b)} x(t) = x(0) + \int_0^t v(s) ds$$

$$= 0 + \int_0^t 100 \tanh(0.1s) ds$$

$$= \frac{1}{0.001} \ln |\cosh(0.1t)| = 1000 \ln |\cosh(0.1t)|$$

Solve  $v(t) = 0.9 \cdot 100$ , we get

$$100 \tanh(0.1t) = 90$$

$$t = 10 \cdot \tanh^{-1} \left( \frac{90}{100} \right)$$

$$\approx 14.7 \text{ s}$$

$$x(14.7) \approx 1000 \cdot \ln |\cosh(0.1 \cdot 14.7)|$$

$$\approx 830 \text{ ft.}$$

Pro. solution:  $32000 \text{ lb} = 1000 \text{ kg}$ .

$$m \frac{dv}{dt} = F$$

$$1000 \frac{dv}{dt} = 5000 - 100v$$

$$\int \frac{1}{5000 - 100v} dv = \int \frac{1}{1000} dt$$

$$-\frac{1}{100} \ln|5000 - 100v| = \frac{t}{1000} + C$$

$$\ln|5000 - 100v| = -\frac{t}{10} + C$$

$$5000 - 100v = A e^{-\frac{t}{10}}$$

$$v(0) = 0 \Rightarrow 5000 = A$$

$$\Rightarrow 5000 - 100v = 5000 e^{-t/10}$$

$$100 v(t) = 5000 - 5000 e^{-t/10}$$

$$v(t) = 50 - 50 e^{-t/10}$$

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} 50 - 50 e^{-t/10} \\ = 50 \text{ ft/s}$$