§2.2.

P1. solution: $\frac{dx}{dt} = x-4.$

$x-4=0 \Rightarrow x=4$ → critical point

$x'<0$, $x'>0$; $x=4$; $x=4$ unstable

$$\int \frac{1}{x-4}\,dx = \int dt.$$

$$\ln|x-4| = t + C$$

$$x-4 = Ae^{t}$$

$x(0)=x_0 \Rightarrow x_0-4 = A \Rightarrow$

$$x-4 = (x_0-4)e^{t}$$

$$x(t) = 4+(x_0-4)e^{t}$$

P3. solution: $\frac{dx}{dt} = x^2-4x$

$x^2-4x=0 \Rightarrow x(x-4)=0$

$\Rightarrow x=0$ or $x=4$

two critical points

$x'>0$, $x'<0$, $x'>0$; $x=0$ stable ; $x=4$ unstable

$$\int \frac{1}{x(x-4)}\,dx = \int dt$$

$$\int \left(\frac{-1}{4}\right)\left(\frac{1}{x} - \frac{1}{x-4}\right) = \int dt.$$

$$\ln|x| - \ln|x-4| = -4t + C$$

$$\frac{x}{x-4} = Ae^{-4t}$$

$x(0)=x_0 \Rightarrow \frac{x_0}{x_0-4} = A$

$$\Rightarrow \frac{x}{x-4} = \frac{x_0e^{-4t}}{x_0-4}$$

$$x(x_0-4) = x_0(x-4)e^{-4t}$$

$$(x_0e^{-4t} - x_0+4)x = 4x_0e^{-4t}$$

$$x(t) = \frac{4x_0e^{-4t}}{x_0e^{-4t}-x_0+4}$$

P4. Solution: $\frac{dx}{dt} = 3x-x^2$

$3x-x^2=0 \Rightarrow x(3-x)=0$

$\Rightarrow x=0$ or $x=3$

$x'<0$, $x'>0$, $x'<0$; $x=0$ unstble ; $x=3$ stable

$$\int \frac{1}{x(3-x)}\,dx = \int dt$$

$$\int \frac{1}{3}\left(\frac{1}{x}+\frac{1}{3-x}\right)dx = t+C$$

$$\ln|x| - \ln|3-x| = 3t+C$$

$$\ln\left|\frac{x}{3-x}\right| = 3t+C$$

$$\frac{x}{3-x} = Ae^{3t}$$

$x(0)=x_0 \Rightarrow A = \frac{x_0}{3-x_0}$

$$\Rightarrow \frac{x}{3-x} = \frac{x_0e^{3t}}{3-x_0}$$

$$x(3-x_0) = x_0(3-x)e^{3t}.$$

$$x(3-x_0+x_0e^{3t}) = 3x_0e^{3t}$$

$$x(t) = \frac{3x_0e^{3t}}{3-x_0+x_0e^{3t}}$$

$$x(t) = \frac{3x_0}{x_0+(3-x_0)e^{-3t}}$$

P6. Solution: $\frac{dx}{dt} = 9-x^2$

$9-x^2=0 \Rightarrow (3-x)(3+x)=0$

$\Rightarrow x=3$ or $x=-3$

two critical points

$x'<0$, $x'>0$, $x'<0$; $x=-3$ unstable ; $x=3$ stable

$$\int \frac{1}{(3-x)(3+x)}\,dx = \int dt.$$

$$\int \frac{1}{6}\left(\frac{1}{3-x}+\frac{1}{3+x}\right)dx = t+C$$

$$-\ln|3-x| + \ln|3+x| = 6t+C$$

$$\ln\left|\frac{3+x}{3-x}\right| = 6t+C$$

$$\frac{3+x}{3-x} = Ae^{6t}$$

$x(0)=x_0 \Rightarrow A = \frac{3+x_0}{3-x_0}$

$$\Rightarrow \frac{3+x}{3-x} = \frac{3+x_0}{3-x_0}e^{6t}$$

$$3(3-x_0)+(3-x_0)x = 3(3+x_0)e^{6t} - x(3+x_0)e^{6t}$$

$$[(3-x_0)+(3+x_0)e^{6t}]x = 3(3+x_0)e^{6t} - 3(3-x_0)$$

$$x(t) = \frac{3(3+x_0)e^{6t} - 3(3-x_0)}{(3+x_0)e^{6t}+(3-x_0)}$$

P7. Solution: $\frac{dx}{dt} = (x-2)^2$.

$(x-2)^2 = 0 \Rightarrow x = 2$ critical point

$x' > 0 \quad x' > 0$

$x = 2$

unstable.

$$\int \frac{1}{(x-2)^2} dx = \int dt.$$

$$-\frac{1}{x-2} = t + C$$

$$x(0) = x_0 \Rightarrow -\frac{1}{x_0-2} = C$$

$$\Rightarrow -\frac{1}{x-2} = t - \frac{1}{x_0-2}$$

$$-1 = t(x-2) - \frac{x-2}{x_0-2}$$

$$(t - \frac{1}{x_0-2})x = 2t - 1 - \frac{2}{x_0-2}$$

$$x = \frac{(2t-1)(x_0-2) - 2}{t(x_0-2) - 1}$$

$$\Rightarrow x(t) = \frac{2tx_0 - 4t - x_0}{tx_0 - 2t - 1}$$

P10. Solution: $\frac{dx}{dt} = 7x - x^2 - 10$

$$7x - x^2 - 10 = 0 \Rightarrow (-x+5)(x-2) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 5$$

$x' < 0 \quad x' > 0 \quad x' < 0$

$x = 2 \qquad x = 5$

unstable stable

$$\int \frac{1}{(-x+5)(x-2)} dx = \int dt.$$

$$\int \frac{1}{3}\left(\frac{1}{-x+5} + \frac{1}{x-2}\right) = t + C$$

$$-\ln|-x+5| + \ln|x-2| = 3t + C$$

$$\ln\left|\frac{x-2}{-x+5}\right| = 3t + C$$

$$\frac{x-2}{-x+5} = Ae^{3t}$$

$$x(0) = x_0 \Rightarrow A = \frac{x_0-2}{-x_0+5}$$

$$\Rightarrow \frac{x-2}{-x+5} = \frac{x_0-2}{-x_0+5} e^{3t}$$

$$(x-2)(-x_0+5) = (-x+5)(x_0-2)e^{3t}$$

$$[-x_0+5+(x_0-2)e^{3t}]x = 2(-x_0+5) + 5(x_0-2)e^{3t}$$

$$x(t) = \frac{2(-x_0+5) + 5(x_0-2)e^{3t}}{-x_0+5+(x_0-2)e^{3t}}$$

§2.3

P1. Solution:

$$\frac{dv}{dt} = k(250 - v)$$

$$\int \frac{1}{250-v} dv = \int k\,dt$$

$$-\ln|250-v| = kt + C$$

$$\ln|250-v| = -kt + C$$

$$250 - v = Ae^{-kt}$$

$$v(t) = 250 - Ae^{-kt}$$

$$v(0) = 0 \Rightarrow 0 = 250 - A \Rightarrow A = 250$$

$$\Rightarrow v(t) = 250 - 250e^{-kt}$$

$$v(10) = 100 \Rightarrow 250 - 250e^{-10k} = 100$$

$$\Rightarrow 250e^{-10k} = 150$$

$$\Rightarrow k = -\frac{1}{10} \cdot \ln\left(\frac{150}{250}\right)$$

$$\approx -0.0511 \Rightarrow v(t) = 250 - 250e^{-0.0511t}$$

To solve $v(t) = 200$, we get

$$250 - 250e^{-0.0511t} = 200$$

$$250e^{-0.0511t} = 50$$

$$t = \left(\frac{1}{-0.0511}\right)\ln\left(\frac{50}{250}\right)$$

$$\approx 31.5\,s$$

P2. Solution:

$$\frac{dv}{dt} = -kv$$

(a) $\int \frac{1}{v} dv = \int -k\,dt$

$$\ln|v| = -kt + C$$

$$v(t) = Ae^{-kt}$$

$$v(0) = v_0 \Rightarrow A = v_0 \Rightarrow v(t) = v_0 e^{-kt}$$

$$x(t) = x_0 + \int_0^t v(s)\,ds$$

$$= x_0 + \int_0^t v_0 e^{-ks} ds$$

$$= x_0 + \left(-\frac{v_0}{k}\right)e^{-ks}\Big|_{s=0}^{s=t} = x_0 + \left(-\frac{v_0}{k}\right)(e^{-kt} - 1)$$

$$= x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt})$$

(b) $\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty}\left[X_0 + \left(\frac{X_0}{k}\right)(1-e^{-kt})\right]$

$$= X_0 + \left(\frac{V_0}{k}\right)\cdot 1$$

$$= X_0 + \frac{V_0}{k}$$

P3. Solution:

$$\left.\begin{array}{l} V(t) = V_0 e^{-kt} \\ V_0 = 40 \end{array}\right\} \Rightarrow V(t) = 40e^{-kt}$$

$$V(10) = 20 \Rightarrow 40e^{-10k} = 20$$

$$\Rightarrow k = \left(\frac{-1}{10}\right)\ln\left(\frac{20}{40}\right)$$

$$\approx 0.0693$$

$$\Rightarrow \lim_{t\to+\infty} X(t) = X_0 + \frac{V_0}{k}$$

$$= \frac{40}{0.0693}$$

$$\approx 577 \text{ ft}.$$

P7. solution:

$$\frac{dv}{dt} = 10 - 0.1v$$

$$\int \frac{1}{10-0.1v}\,dv = \int dt$$

$$-10\ln|10-0.1v| = t + C$$

$$\ln|10-0.1v| = -\frac{t}{10} + C$$

$$10 - 0.1v = Ae^{-t/10}$$

$$v(0) = 0 \Rightarrow 10 = A \Rightarrow$$

$$10 - 0.1v = 10e^{-t/10}$$

$$0.1v = 10 - 10e^{-t/10}$$

$$v(t) = 100 - 100e^{-t/10}$$

(a) $\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty}[100 - 100e^{-t/10}]$

$$= 100 \text{ ft/s}$$

(b) Solve $v(t) = 0.9\cdot 100$, we get

$$100 - 100e^{-t/10} = 90$$

$$-100e^{-t/10} = -10$$

$$t = (-10)\ln\left(\frac{10}{100}\right)$$

$$\approx 23 \text{ s}$$

$$X(t) = X(0) + \int_0^t v(s)\,ds$$

$$= 0 + \int_0^t (100 - 100e^{-s/10})\,ds$$

$$= (100s + 1000e^{-s/10})\Big|_{s=0}^{s=t}$$

$$= 100t + 1000(e^{-t/10} - 1)$$

$$X(23) = 23\cdot 100 + 1000(e^{-23/10} - 1)$$

$$= 2300 + 1000(-0.8997)$$

$$\approx 1400.26 \text{ ft}$$

P8. Solution:

$$\frac{dv}{dt} = 10 - 0.001v^2$$

$$\int \frac{1}{10-0.001v^2}\,dv = \int dt$$

$$v(t) = \sqrt{\frac{10}{0.001}}\tanh\left(C - \sqrt{10(0.001)}\,t\right)$$

$$= 100\tanh(0.1t - C)$$

$$v(0) = 0 \Rightarrow 0 = 100\tanh(-C)$$

$$\Rightarrow C = 0$$

$$\Rightarrow v(t) = 100\tanh(0.1t)$$

(a) $\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} 100\tanh(0.1t)$

$$= 100$$

(b) $X(t) = X(0) + \int_0^t v(s)\,ds$

$$= 0 + \int_0^t 100\tanh(0.1s)\,ds$$

$$= \frac{1}{0.001}\ln|\cosh(0.1t)| = 1000\ln|\cosh(0.1t)|$$

Solve $v(t) = 0.9\cdot 100$, we get

$$100\tanh(0.1t) = 90$$

$$t = 10\cdot\tanh^{-1}\left(\frac{90}{100}\right)$$

$$\approx 14.7 \text{ s}$$

$$X(14.7) \approx 1000\cdot\ln|\cosh(0.1\times 14.7)|$$

$$\approx 830 \text{ ft}.$$

P10. solution: $32000\,lb = 1000\,kg$.

$$m\frac{dv}{dt} = F$$

$$1000\frac{dV}{dt} = 5000 - 100V$$

$$\int \frac{1}{5000-100V}\,dv = \int \frac{1}{1000}\,dt$$

$$-\frac{1}{100}\ln|5000-100V| = \frac{t}{1000} + C$$

$$\ln|5000-100V| = -\frac{t}{10} + C$$

$$5000 - 100V = Ae^{-\frac{t}{10}}$$

$$V(0) = 0 \Rightarrow 5000 = A$$

$$\Rightarrow 5000 - 100V = 5000e^{-t/10}$$

$$100\,V(t) = 5000 - 5000e^{-t/10}$$

$$V(t) = 50 - 50e^{-t/10}$$

$$\lim_{t\to+\infty} V(t) = \lim_{t\to+\infty} 50 - 50e^{-t/10}$$

$$= 50 \text{ ft/s}$$