§2.1

P1. Solution: $\frac{dx}{dt} = x - x^2$, $x(0) = 2$

$$\frac{dx}{dt} = x(1-x)$$

$$\int \frac{1}{x(1-x)}\,dx = \int dt$$

$$\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = t + C$$

$$\ln|x| - \ln|1-x| = t + C$$

$$\ln\left|\frac{x}{1-x}\right| = t + C$$

$$\frac{x}{1-x} = Ae^{t}$$

$$x(0) = 2 \Rightarrow \frac{2}{1-2} = A$$

$$A = -2$$

Thus,

$$\frac{x}{1-x} = -2e^{t}$$

$$x = -2(1-x)e^{t}$$

$$(2e^{t}-1)x = 2e^{t} \Rightarrow x(t) = \frac{2e^{t}}{2e^{t}-1} = \frac{2}{2-e^{-t}}$$

P3. Solution: $\frac{dx}{dt} = 1 - x^2$, $x(0) = 3$

$$\frac{dx}{dt} = (1-x)(1+x)$$

$$\int \frac{1}{(1-x)(1+x)}\,dx = \int dt$$

$$\int \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx = t + C$$

$$\frac{1}{2}\left(-\ln|1-x| + \ln|1+x|\right) = t + C$$

$$\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = t + C$$

$$\ln\frac{1+x}{1-x} = 2t + C$$

$$\frac{1+x}{1-x} = Ae^{2t}$$

$$x(0) = 3 \Rightarrow \frac{1+3}{1-3} = A \Rightarrow A = -2$$

Thus, $\frac{1+x}{1-x} = -2e^{2t}$

$$1+x = (1-x)(-2)e^{2t}$$

$$(2e^{2t}-1)x = 2e^{2t}+1$$

$$x(t) = \frac{2e^{2t}+1}{2e^{2t}-1} = \frac{2+e^{-2t}}{2-e^{-2t}}$$

P6. Solution: $\frac{dx}{dt} = 3x(x-5)$, $x(0) = 2$

$$\int \frac{1}{x(x-5)}\,dx = \int 3\,dt$$

$$\int \left(-\frac{1}{5}\right)\left(\frac{1}{x} - \frac{1}{x-5}\right) dx = 3t + C$$

$$-\frac{1}{5}\left(\ln|x| - \ln|x-5|\right) = 3t + C$$

$$\ln\left|\frac{x}{x-5}\right| = -15t + C$$

$$\frac{x}{x-5} = Ae^{-15t}$$

$$x(0) = 2 \Rightarrow \frac{2}{2-5} = A \Rightarrow A = -\frac{2}{3}$$

Thus, $\frac{x}{x-5} = -\frac{2}{3}e^{-15t}$

$$3x = -2(x-5)e^{-15t}$$

$$(2e^{-15t}+3)x = 10e^{-15t}$$

$$x(t) = \frac{10e^{-15t}}{2e^{-15t}+3}$$

P8. Solution: $\frac{dx}{dt} = 7x(x-13)$, $x(0) = 17$.

$$\int \frac{1}{x(x-13)}\,dx = \int 7\,dt$$

$$\int -\frac{1}{13}\left(\frac{1}{x} - \frac{1}{x-13}\right) dx = 7t + C$$

$$-\frac{1}{13}\left(\ln|x| - \ln|x-13|\right) = 7t + C.$$

$$\ln\left|\frac{x}{x-13}\right| = -91t + C$$

$$\frac{x}{x-13} = Ae^{-91t}$$

$$x(0) = 17 \Rightarrow \frac{17}{17-13} = A \Rightarrow A = \frac{17}{4}$$

Thus, $\frac{x}{x-13} = \frac{17}{4}e^{-91t}$

$$4x = 17(x-13)e^{-91t}$$

$$(17e^{-91t}-4)x = 221e^{-91t}$$

$$x(t) = \frac{221e^{-91t}}{17e^{-91t}-4} = \frac{221}{17-4e^{+91t}}$$

P9. Solution:

$$\frac{dP}{dt} = k\sqrt{P}$$

$$\int \frac{1}{\sqrt{P}}\,dP = \int k\,dt \Rightarrow P(t) = \left(\frac{k}{2}t + C\right)^2$$

$$2\sqrt{P} = kt + C$$

we need to determine two constants k, C.

$P(0)=100 \Rightarrow 100=C^2 \Rightarrow C=10$

$$\left.\frac{dP}{dt}\right|_{t=0}=20 \Rightarrow k\sqrt{P(0)}=20$$
$$\Rightarrow k=\frac{20}{\sqrt{P(0)}}=2$$

Thus. $P(t)=(t+10)^2$

$P(12)=(12+10)^2=22^2=484.$

P15. Solution:

$$\frac{dP}{dt}=aP-bP^2$$
$$=kP(M-P)$$

where $k=b,\ M=\frac{a}{b}$.

$$\int\frac{1}{P(M-P)}dP=\int k\,dt$$
$$\int\frac{1}{M}\left(\frac{1}{P}+\frac{1}{M-P}\right)=\int k\,dt$$
$$\frac{1}{M}(\ln|P|-\ln|M-P|)=kt+C$$
$$\ln\left|\frac{P}{M-P}\right|=Mkt+C$$
$$\frac{P}{M-P}=Ae^{Mkt}$$
$$P(0)=P_0 \Rightarrow A=\frac{P_0}{M-P_0}$$

Thus, $\frac{P}{M-P}=\frac{P_0}{M-P_0}e^{Mkt}$

$$\Rightarrow P(t)=\frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$$

$$\left.\begin{array}{l} aP_0=B_0 \Rightarrow a=\frac{B_0}{P_0} \\ bP_0^2=D_0 \Rightarrow b=\frac{D_0}{P_0^2}\end{array}\right\} \Rightarrow M=\frac{B_0}{\cancel{P_0}}\cdot\frac{P_0^{\cancel{2}}}{D_0}=\frac{B_0P_0}{D_0}$$

Thus

$$\lim_{t\to+\infty}P(t)=\lim_{t\to+\infty}\frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$$
$$=\frac{MP_0}{P_0}$$
$$=M$$
$$=\frac{B_0P_0}{D_0}$$

P16. Solution:

$P_0=120,\ B_0=8,\ D_0=6$

$$\Rightarrow M=\frac{B_0P_0}{D_0}=\frac{8\times120}{6}=160.$$
$$k=b=\frac{D_0}{P_0^2}=\frac{6}{(120)^2}=\frac{1}{2400}$$

Thus,

$$P(t)=\frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$$
$$=\frac{160\cdot120}{120+(160-120)e^{-\frac{160}{2400}t}}$$
$$=\frac{480}{3+e^{-t/15}}$$

we need to solve t from $P(t)=0.95M$

$$\frac{\cancel{480}\,3}{3+e^{-t/15}}=0.95\times\cancel{160}$$
$$0.95e^{-t/15}=0.15$$
$$t=(-15)\ln\left(\frac{0.15}{0.95}\right)$$
$$\approx 27.69 \text{ months}$$

P19. Solution: $\frac{dP}{dt}=kP(P-M)$, $P(0)=P_0$

$$\Rightarrow P(t)=\frac{MP_0}{P_0+(M-P_0)e^{kMt}}$$

$P_0=100,$ | $9=\delta P_0$ $\Rightarrow \delta=0.09$ | $10=kP_0^2 \Rightarrow$ $10=k\cdot100^2 \Rightarrow k=\frac{1}{1000}$

$$\Rightarrow M=\frac{\delta}{k}=\frac{0.09}{\left(\frac{1}{1000}\right)}=90$$
$$\Rightarrow P(t)=\frac{90\cdot100}{100+(90-100)e^{0.09t}}$$
$$=\frac{90}{1\ \ 0.1e^{0.09t}}$$

we need to solve t from $P(t)=10M$

$$\frac{\cancel{90}}{1-0.1e^{0.09t}}=10\cdot\cancel{90}$$
$$1-0.1e^{0.09t}=0.1$$
$$e^{0.09t}=\frac{1}{0.09}\ln 9$$
$$\approx 24.41 \text{ months}$$