§1.6

P.2. Solution: $2xyy' = x^2 + 2y^2$

$$y' = \frac{x}{2y} + \frac{y}{x}$$

$$v = \frac{y}{x},\ y = vx,\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1}{2v} + v$$

$$x\frac{dv}{dx} = \frac{1}{2v}$$

$$\int 2v\,dv = \int \frac{1}{x}dx$$

$$v^2 = \ln|x| + C$$

$$\frac{y^2}{x^2} = \ln|x| + C$$

$$y^2 = x^2\ln|x| + Cx^2$$

P4. Solution: $(x-y)y' = x+y$.

$$y' = \frac{x+y}{x-y}$$

$$\Rightarrow y' = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$v = \frac{y}{x},\ y = vx,\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$\int \frac{1-v}{1+v^2}dv = \int \frac{1}{x}dx$$

$$\tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + C$$

P7. Solution: $xy^2y' = x^3 + y^3$

$$y' = \left(\frac{x}{y}\right)^2 + \frac{y}{x}$$

$$v = \frac{y}{x},\ y = vx,\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{v} + x\frac{dv}{dx} = \frac{1}{v^2} + \cancel{v}$$

$$x\frac{dv}{dx} = \frac{1}{v^2}$$

$$\int v^2\,dv = \int \frac{1}{x}dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$y^3/x^3 = 3(\ln|x| + C)$$

$$y^3 = 3x^3(\ln|x| + C)$$

P12. Solution: $xyy' = y^2 + x\sqrt{4x^2+y^2}$

$$y' = \frac{y}{x} + \sqrt{4\left(\frac{x}{y}\right)^2 + 1}$$

$$v = \frac{y}{x},\ y = vx,\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v + \sqrt{\frac{4}{v^2} + 1}$$

$$x\frac{dv}{dx} = \sqrt{\frac{4}{v^2} + 1} = \frac{\sqrt{4+v^2}}{v}$$

$$\int \frac{v}{\sqrt{4+v^2}}dv = \int \frac{1}{x}dx$$

$$\sqrt{4+v^2} = \ln|x| + C$$

$$\sqrt{4+\frac{y^2}{x^2}} = \ln|x| + C$$

$$x\sqrt{4x^2+y^2} = \ln|x| + C$$

P16. Solution: $y' = \sqrt{x+y+1}$

$$v = \sqrt{x+y+1},\ y = v^2 - x - 1,\ \frac{dy}{dx} = -1 + 2v\frac{dv}{dx}$$

$$\Rightarrow -1 + 2v\frac{dv}{dx} = v.$$

$$2v\frac{dv}{dx} = v + 1$$

$2 - \frac{2}{v+1}$

$$\int \frac{2v}{v+1}dv = \int dx$$

$$\int 2 - \frac{2}{v+1}dv = \int dx$$

$$2v - \ln(v+1)^2 = x + C$$

$$2\sqrt{x+y+1} + \ln(\sqrt{x+y}+1)^2 = x + C$$

P17. Solution: $y' = (4x+y)^2$

$$v = 4x+y,\ y = v - 4x,\ \frac{dy}{dx} = -4 + \frac{dv}{dx}$$

$$\Rightarrow -4 + \frac{dv}{dx} = v^2$$

$$\frac{dv}{dx} = v^2 + 4$$

$$\int \frac{1}{v^2+4}dv = \int dx$$

using $x = 2v$

$$\frac{1}{2}\tan^{-1}(v/2) = x + C$$

$$\frac{v}{2} = \tan(2x + A)$$

$$v = 2\tan(2x + A) \Rightarrow y = 4x + 2\tan(2x+A)$$

P22. Solution: $x^2y' + 2xy = 5y^4$

$$y' + \frac{2}{x}y = \frac{5}{x^2}y^4 \quad \rightarrow n = 4,\ 1-n = -3$$

$$v = y^{-3},\ y = v^{-\frac{1}{3}},\ \frac{dy}{dx} = -\frac{1}{3}v^{-\frac{4}{3}}\frac{dv}{dx}$$

$$-\frac{1}{3}v^{-\frac{4}{3}}\frac{dv}{dx} + \frac{2}{x}v^{-\frac{1}{3}} = \frac{5}{x^2}v^{-\frac{4}{3}}$$

$$\frac{dv}{dx} - \frac{6}{x}v = \frac{-15}{x^2}$$

$P(x) = -\frac{6}{x}$, $Q(x) = -\frac{15}{x^2}$

$$\Rightarrow \rho(x) = e^{\int P(x)dx} = e^{-\int \frac{6}{x}dx}$$

$$= e^{-6\ln|x|} = x^{-6}$$

$$\Rightarrow x^{-6}\frac{dv}{dx} - \frac{6}{x^7}v = \frac{-15}{x^8}$$

$$\frac{d}{dx}[x^{-6}v] = \frac{-15}{x^8}$$

$$x^{-6}v = \int \frac{-15}{x^8}dx = \frac{15}{7}x^{-7}$$

$$\Rightarrow v(x) = \frac{15}{7x} + C$$

Thus $\left(\frac{1}{y(x)}\right)^3 = \frac{15}{7x} + C$

$$\Rightarrow y(x) = \left(\frac{15}{7x} + C\right)^{-\frac{1}{3}}$$

P26. solution: $3y^2y' + y^3 = e^{-x}$

$$y' + \frac{1}{3}y = \frac{1}{3}e^{-x}y^{-2} \quad (n=-2)$$

$$v = y^{1-n} = y^3,\ y = v^{\frac{1}{3}},\ \frac{dy}{dx} = \frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx}$$

$$\Rightarrow \frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx} + \frac{1}{3}v^{\frac{1}{3}} = \frac{1}{3}e^{-x}v^{-\frac{2}{3}}$$

$$\frac{dv}{dx} + v = e^{-x}$$

$\Rightarrow P(x) = 1 \quad Q(x) = e^{-x}$

$$\Rightarrow \rho(x) = e^{\int P(x)dx} = e^{x}$$

$$e^x\frac{dv}{dx} + e^x v = e^x e^{-x}$$

$$\frac{d}{dx}[e^x v] = 1$$

$$e^x v = \int 1\,dx$$

$$= x + C$$

$$\Rightarrow v(x) = e^{-x}(x+C)$$

$$y(x) = (v(x))^{\frac{1}{3}}$$

$$= e^{-\frac{x}{3}}(x+c)^{\frac{1}{3}}$$

P31. Solution: $(2x+3y)dx + (3x+2y)dy = 0$

$M(x,y) = 2x+3y$, $N(x,y) = 3x+2y$

$\frac{\partial M}{\partial y} = 3$, $\frac{\partial N}{\partial x} = 3$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ exact!

$\frac{\partial F}{\partial x} = M(x,y) \Rightarrow$

$$F(x,y) = \int M(x,y)dx = \int (2x+3y)dx$$

$$= x^2 + 3xy + g(y)$$

$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow$

$$\frac{\partial F}{\partial y} = 3x + g'(y) = 3x + 2y$$

$$\Rightarrow g'(y) = 2y$$

$$\Rightarrow g(y) = \int 2y\,dy = y^2 + C_1$$

Thus $F(x,y) = x^2 + 3xy + y^2 + C_1$

and a general solution is implicitly difined by $x^2 + 3xy + y^2 = C$.

P36. solution: $(1+ye^{xy})dx + (2y + xe^{xy})dy = 0$

$M(x,y) = 1 + ye^{xy}$, $N(x,y) = 2y + xe^{xy}$

$\frac{\partial M}{\partial y} = xy\,e^{xy}$, $\frac{\partial N}{\partial x} = xy\,e^{xy}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ exact!

$\frac{\partial F}{\partial x} = M(x,y) \Rightarrow$

$$F(x,y) = \int M(x,y)dx = \int (1+ye^{xy})dx$$

$$= x + e^{xy} + g(y)$$

$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow$

$$\frac{\partial F}{\partial y} = xe^{xy} + g'(y) = 2y + xe^{xy}$$

$$\Rightarrow g'(y) = 2y$$

$$\Rightarrow g(y) = \int g'(y)dy = \int 2y\,dy = y^2 + C_1$$

Thus, $F(x,y) = x + e^{xy} + y^2 + C_1$

and a general solution is implicitly given by $x + e^{xy} + y^2 = C$.

P40. solution:

$$(e^x\sin y + \tan y)dx + (e^x\cos y + x\sec^2 y)dy = 0$$

$$M(x,y) = e^x\sin y + \tan y$$
$$N(x,y) = e^x\cos y + x\sec^2 y$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = e^x\cos y + \sec^2 y \\ \frac{\partial N}{\partial x} = e^x\cos y + \sec^2 y\end{array}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact!}$$

$$\frac{\partial F}{\partial x} = M(x,y) \Rightarrow$$
$$F(x,y) = \int M(x,y)dx$$
$$= \int e^x\sin y + \tan y \, dx$$
$$= e^x\sin y + x\tan y + g(y)$$

$$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow$$
$$\frac{\partial F}{\partial y} = e^x\cos y + x\sec^2 y + g'(y)$$
$$= e^x\cos y + x\sec^2 y$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

Thus
$$F(x,y) = e^x\sin y + x\tan y + C_1$$

and a general solution is given by

$$e^x\sin y + x\tan y = C \quad (\text{take } C_1 = 0)$$