§1.5.

P1. Solution: $y'+y=2,\ y(0)=0$

$P(x)=1,\ Q(x)=2.$

Integrating factor:

$\rho(x)=e^{\int P(x)dx}=e^{\int 1dx}=e^{x}.$

Thus,

$e^{x}y'+e^{x}y=2e^{x}$

$\frac{d}{dx}[e^{x}y]=2e^{x}$

$e^{x}y=\int 2e^{x}dx$

$=2e^{x}+C$

$\Rightarrow y(x)=e^{-x}(2e^{x}+c)$

$\boxed{y(x)=2+ce^{-x}}$

$y(0)=0 \Rightarrow 0=2+C\cdot e^{0}$

$\Rightarrow C=-2$

Thus, the particular solution is

$\boxed{y(x)=2-2e^{-x}}$

P3. solution: $y'+3y=2xe^{-3x}$

$P(x)=3,\ Q(x)=2xe^{-3x}$

Integrating factor:

$\rho(x)=e^{\int P(x)dx}=e^{\int 3dx}=e^{3x}$

Thus

$e^{3x}y'+3e^{3x}y=2xe^{-3x}\cdot e^{3x}$

$\frac{d}{dx}[e^{3x}y]=2x$

$e^{3x}y=\int 2x\,dx$

$=x^2+C$

$\Rightarrow \boxed{y(x)=e^{-3x}(x^2+C)}$

P4. Solution: $y'-2xy=e^{x^2}$

$P(x)=-2x,\ Q(x)=e^{x^2}$

Integrating factor:

$\rho(x)=e^{\int P(x)dx}=e^{\int -2xdx}=e^{-x^2}$

Thus

$e^{-x^2}y'-(e^{-x^2})2xy=e^{x^2}e^{-x^2}$

$\frac{d}{dx}[e^{-x^2}y]=1$

$e^{-x^2}y=\int 1dx=x+C$

$\Rightarrow \boxed{y(x)=e^{x^2}(x+C)}$

P6. Solution: $xy'+5y=7x^2,\ y(2)=5$

standard form: $y'+\frac{5y}{x}=7x$

$P(x)=\frac{5}{x},\ Q(x)=7x$

Integrating factor:

$\rho(x)=e^{\int P(x)dx}=e^{\int \frac{5}{x}dx}=e^{5\ln x}=x^5$

Thus

$x^5y'+5x^4y=7x\cdot x^5$

$\frac{d}{dx}[x^5y]=7x^6$

$x^5y=\int 7x^6dx=x^7+C$

$\Rightarrow y(x)=x^{-5}(x^7+c)$

$\boxed{y(x)=x^2+cx^{-5}}$

$y(2)=5 \Rightarrow 5=2^2+C2^{-5}$

$\Rightarrow C=2^5$

Thus, the particular solution is

$$\boxed{\begin{aligned} y(x)&=x^2+2^5x^{-5}\\ &=x^2+\left(\tfrac{2}{x}\right)^5\end{aligned}}$$

P11. Solution: $xy'+y=3xy,\ y(1)=0$

standard form: $y'+\frac{y}{x}=3y$

$y'+(\frac{1}{x}-3)y=0$

$P(x) = \frac{1}{x} - 3$, $Q(x) = 0 \rightarrow$ seperable

$$\boxed{y(x)} = ce^{-\int P(x)dx} = ce^{-\int \frac{1}{x} - 3dx}$$

$$\boxed{= ce^{-\ln x + 3x}}$$

$$y(1) = 0 \Rightarrow 0 = ce^{-\ln 1 + 3}$$

$$0 = ce^3$$

$$\Rightarrow c = 0$$

Thus, the particular solution is

$$\boxed{y(x) \equiv 0}$$

P17. Solution: $(1+x)y' + y = \cos x$, $y(0) = 1$

Standard form: $y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}$

$$P(x) = \frac{1}{1+x},\quad Q(x) = \frac{\cos x}{1+x}$$

Integrating factor:

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)}$$

$$= 1+x$$

Thus,

$$(1+x)y' + y = \cos x$$

$$\frac{d}{dx}[(1+x)y] = \cos x$$

$$(1+x)y = \int \cos x\,dx = \sin x + C$$

$$\Rightarrow \boxed{y(x) = \frac{1}{1+x}(\sin x + C)}$$

$$y(0) = 1 \Rightarrow 1 = \frac{1}{1}(\sin 0 + C)$$

$$\Rightarrow C = 1$$

Thus, the particular solution is

$$y(x) = \frac{\sin x + 1}{1+x}$$

P19. Solution: $y' + y\cot x = \cos x$

$$P(x) = \cot x,\quad Q(x) = \cos x$$

Integrating factor

$$\rho(x) = e^{\int P(x)dx} = e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$$

Thus,

$$\sin x\, y' + \sin x \cdot \cot x\, y = \cos x \cdot \sin x$$

$$\frac{d}{dx}[\sin x\, y] = \cos x \sin x.$$

$$\sin x\, y = \int \cos x \sin x\, dx$$

$$= \frac{\sin^2 x}{2} + C$$

$$\Rightarrow \boxed{y(x) = \frac{1}{\sin x}\left(\frac{\sin^2 x}{2} + C\right) = \frac{\sin x}{2} + \frac{C}{\sin x}}$$

P.21. Solution: $xy' = 3y + x^4\cos x$, $y(2\pi) = 0$.

Standard form: $y' - \frac{3}{x}y = x^3\cos x$

$$P(x) = -\frac{3}{x},\quad Q(x) = x^3\cos x$$

Integrating factor:

$$\rho(x) = e^{\int P(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$$

Thus

$$\frac{1}{x^3}y' - \frac{3}{x^4}y = \cos x$$

$$\frac{d}{dx}\left[\frac{1}{x^3}y\right] = \cos x$$

$$\frac{1}{x^3}y = \int \cos x\, dx = \sin x + C$$

$$\Rightarrow \boxed{y(x) = x^3(\sin x + C)}$$

$$y(2\pi) = 0 \Rightarrow 0 = (2\pi)^3(\sin 2\pi + C)$$

$$\Rightarrow C = 0$$

Thus, the particular solution is

$$y(x) = x^3\sin x$$

P24. Solution: $(x^2+4)y' + 3xy = x$, $y(0) = 1$

Standard form: $y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$

$$P(x) = \frac{3x}{x^2+4},\quad Q(x) = \frac{x}{x^2+4}$$

Integrating factor

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{3x}{x^2+4}dx}$$

$$= e^{\frac{3}{2}\ln(x^2+4)} = (x^2+4)^{3/2}$$

Thus

$$(x^2+4)^{\frac{3}{2}} y' + (x^2+4)^{\frac{1}{2}}(3x) y = x(x^2+4)^{\frac{1}{2}}$$

$$\frac{d}{dx}\left[(x^2+4)^{\frac{3}{2}} y\right] = x(x^2+4)^{\frac{1}{2}}$$

$$(x^2+4)^{\frac{3}{2}} y = \int x(x^2+4)^{\frac{1}{2}}\, dx$$

$$= \frac{1}{3}(x^2+4)^{\frac{3}{2}} + C$$

$$\Rightarrow \boxed{y(x) = \frac{1}{3} + C(x^2+4)^{-3/2}}$$

$$y(0) = 1 \Rightarrow 1 = \frac{1}{3} + C \cdot 4^{-3/2}$$

$$\Rightarrow C = \frac{16}{3}$$

Thus, the particular solution is

$$y(x) = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}$$

P30. Solution: $2x\frac{dy}{dx} = y + 2x\cos x$, $y(1) = 0$

standard form: $\frac{dy}{dx} - \frac{1}{2x} y = \cos x$

$P(x) = -\frac{1}{2x}$, $Q(x) = \cos x$

Integrating factor:

$$\rho(x) = e^{\int_{x_0}^{x} P(t)dt} = e^{\int_1^x -\frac{1}{2t}dt}$$

$$= e^{-\frac{1}{2}\ln t \big|_{t=1}^{x}} = e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}$$

Thus

$$\boxed{y(x)} = \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(t)Q(t)dt\right]$$

$$= \sqrt{x}\left[0 + \int_1^x \frac{1}{\sqrt{t}} \cdot \cos t\, dt\right]$$

$$\boxed{= \sqrt{x}\int_1^x \frac{\cos t}{\sqrt{t}}\, dt}$$

P34. Solution: $V_0 = V(0) = 8000$ million $ft^3$

$x_0 = x(0) = 8000 \cdot 0.25\% = 20$

Inflow: $r_i = 500$, $c_i = 0.0005$

outflow: $r_o = 500$, $c_o = \frac{x(t)}{V(t)}$

$V(t) = V_0 + (r_i - r_o)t = V_0 = 8000$

Thus, $c_o = \frac{x(t)}{V_0} = \frac{x(t)}{8000}$.

The equation is then

$$\frac{dx}{dt} + \frac{r_o}{V} x = r_i c_i$$

$$\frac{dx}{dt} + \frac{500}{8000} x = 500 \cdot 0.0005$$

$$\begin{cases} \frac{dx}{dt} + \frac{1}{16} x = 0.25 \\ x(0) = 20. \end{cases}$$

$P(t) = \frac{1}{16}$, $Q(t) = 0.25$

Integrating factor is

$$\rho(x) = e^{\int P(x)d} = e^{\frac{t}{16}}$$

Thus.

$$\frac{d}{dt}\left[e^{\frac{t}{16}} x\right] = 0.25e^{t/16}$$

$$e^{t/16} x = \int 0.25 e^{t/16} dt = 4e^{t/16} + C$$

$$\Rightarrow x(t) = e^{-\frac{t}{16}}(4e^{t/16} + C)$$

$x(0) = 20 \Rightarrow C = 16$, then we have

$$x(t) = e^{-\frac{t}{16}}(4e^{\frac{t}{16}} + 16)$$

Reducing the pollutant concentration to 0.10% means at time $t$

$$x(t) = 8000 \cdot 0.10\% = 8$$

We solve

$$8 = e^{-\frac{t}{16}}(4e^{\frac{t}{16}} + 16) = 4 + 16e^{-\frac{t}{16}}$$

$$\Rightarrow t \approx 22.2 \text{ days}$$

P33. Solution.

$V_0 = V(0) = 1000$ liters

$x_0 = x(0) = 100$ kg

Inflow: $r_i = 5$ L/s, $c_i = 0$

outflow: $r_o = 5$ L/s, $c_o = \frac{x(t)}{V(t)}$

$V(t) = V_0 + (r_i - r_o)t = V_0 = 1000$

$$\Rightarrow c_o = \frac{x(t)}{V_0} = \frac{x(t)}{1000}$$

The equation is then

$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i C_i$$

$$\frac{dx}{dt} + \frac{5}{1000}X = 5 \cdot 0$$

$$\frac{dx}{dt} \quad \frac{1}{200}X = 0$$

$$\Rightarrow \quad X(t) = Ce^{-\int P(x)dx}$$

$$= Ce^{-\frac{t}{200}}$$

$$X(0) = 100 \Rightarrow C = 100$$

$$\Rightarrow x(t) = 100e^{-\frac{t}{200}}$$

We solve $t$ for $X(t) = 10$

$$10 = 100\, e^{-\frac{t}{200}}$$

$$\frac{1}{10} = e^{-\frac{t}{200}}$$

$$t = -200 \cdot \ln\left(\frac{1}{10}\right)$$

$$\approx 460.52 \text{ seconds}$$

$$\approx 7 \text{ min } 41 \text{ secs}$$