§1.3.

P11. Solution:

$f(x,y) = 2x^2y^2$
$\frac{\partial f}{\partial y}(x,y) = 4x^2y$ } continuous anywhere.

Theorem 1 does guarantee a unique solution to $\frac{dy}{dx} = 2x^2y^2$, $y(1) = -1$. near $x=1$.

P12. solution:

$f(x,y) = x\ln y$ is continuous if $y>0$

$\frac{\partial f}{\partial y}(x,y) = \frac{x}{y}$ is continuous if $y \neq 0$

$\Rightarrow f(x,y), \frac{\partial f}{\partial y}(x,y)$ are continuous near $(1,1)$

Thus, Theorem 1 does guarantee a unique solution to $\frac{\partial y}{\partial x} = x\ln y$, $y(1)=1$. near $x=1$.

P13/14. Solution: $f(x,y) = \sqrt[3]{y}$ is continuous everywhere.

$\frac{\partial f}{\partial y}(x,y) = \frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{3y^{\frac{2}{3}}}$ is continuous if $y \neq 0$

i) when $y(0)=1$, Theorem 1 does guarantee a unique solution. near $x=0$

ii) when $y(0)=0$, Theorem 1 does guarantee the existence of a solution near $x=0$, but the solution may not be unique.

P19. Solution:

$f(x,y) = \ln(1+y^2)$ is continuous everywhere.

$\frac{\partial f}{\partial y}(x,y) = \frac{2y}{1+y^2}$ is continuous everywhere.

Theorem 1 does guarantee a unique solution near $x=0$ to $\frac{dy}{dx} = \ln(1+y^2)$. $y(0)=0$

P18. solution:

$f(x,y) = \frac{x-1}{y}$ is continuous when $y \neq 0$.

$\frac{\partial f}{\partial y}(x,y) = -\frac{x-1}{y^2}$ is continuous when $y \neq 0$.

$\Rightarrow \frac{\partial f}{\partial y}, f$ are not continuous on $(1,0)$

Thus, Theorem doesn't guarantee a solution near $x=1$.

P25. Solution:

The limiting velocity will be.

$32 - 1.6v = 0 \Rightarrow v = \frac{32}{1.6}$
$= 20$ ft/sec.

§1.4.

P3. Solution:

$\frac{dy}{dx} = y\sin x$

$\int \frac{1}{y}\,dy = \int \sin x\,dx$

$\ln|y| = -\cos x + C$

$|y| = e^{-\cos x + C} = e^C e^{-\cos x}$
$= Ae^{-\cos x}$ $(A>0)$

[or $y(x) = Ce^{-\cos x}$]

P4. Solution:

$(1+x)\frac{dy}{dx} = 4y$

$\int \frac{1}{y}\,dy = \int \frac{4}{1+x}\,dx$

$\ln|y| = 4\ln|1+x| + C$

$|y| = Ae^{4\ln|1+x|}$
$= A(1+x)^4$ $(A>0)$

[or $y(x) = C(1+x)^4$]

P6. Solution:

$\frac{dy}{dx} = 3\sqrt{xy}$

$\int \frac{1}{\sqrt{y}}\,dy = \int 3\sqrt{x}\,dx$

$$2\sqrt{y} = 2x^{\frac{3}{2}} + C$$
$$\sqrt{y} = x^{\frac{3}{2}} + C$$
$$y(x) = (x^{\frac{3}{2}} + C)^2$$

P.9 Solution:

$$(1-x^2)\frac{dy}{dx} = 2y$$
$$\int \frac{1}{y}\,dy = \int \frac{2}{1-x^2}\,dx$$
$$\ln|y| = \ln\left|\frac{x+1}{x-1}\right| + C$$
$$|y| = A e^{\ln\left|\frac{x+1}{x-1}\right|}$$
$$|y| = A\left|\frac{x+1}{x-1}\right| \quad (A>0)$$
$$\left[\text{or } y(x) = C\frac{x+1}{x-1}\right]$$

P12. Solution:

$$yy' = x(y^2+1)$$
$$\int \frac{y}{y^2+1}\,dy = \int x\,dx$$
$$\frac{1}{2}\ln(y^2+1) = \frac{x^2}{2} + C$$
$$\ln(y^2+1) = x^2 + C$$
$$y^2+1 = Ae^{x^2}$$
$$y^2 = Ae^{x^2} - 1 \quad (A>0)$$

P17. Solution:

$$y' = 1 + x + y + xy.$$
$$\frac{dy}{dx} = (1+x)(1+y)$$
$$\int \frac{1}{1+y}\,dy = \int (1+x)\,dx$$
$$\ln|1+y| = x + \frac{x^2}{2} + C$$
$$|1+y| = Ae^{x+x^2/2} \quad (A>0)$$
$$\left[\text{or } y(x) = Ce^{x+x^2/2} - 1\right]$$

P19. Solution:

$$\frac{dy}{dx} = ye^x, \quad y(0) = 2e.$$
$$\int \frac{1}{y}\,dy = \int e^x\,dx$$
$$\ln|y| = e^x + C$$
$$\Rightarrow |y| = Ae^{e^x} \quad (A>0)$$
$$\Rightarrow y = Ae^{e^x} \text{ or } y = -Ae^{e^x}$$
$$y(0) = 2e > 0 \Rightarrow y(x) = Ae^{e^x}$$
$$Ae = 2e \Rightarrow A = 2$$

Thus, the solution is

$$y(x) = 2e^{e^x}$$

P21. Solution:

$$2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \quad y(5) = 2$$
$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx.$$
$$y^2 = \sqrt{x^2-16} + C$$
$$y(5) = 2 \Rightarrow 2^2 = \sqrt{5^2-16} + C$$
$$\Rightarrow C = 1$$

Thus the solution is

$$y^2 - \sqrt{x^2-16} - 1 = 0$$

P24. Solution:

$$(\tan x)\frac{dy}{dx} = y, \quad y(\tfrac{1}{2}\pi) = \tfrac{1}{2}\pi.$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{\tan x}\,dx$$
$$\ln|y| = \int \frac{\cos x}{\sin x}\,dx + C$$
$$= \ln|\sin x| + C$$
$$\Rightarrow |y| = e^{\ln|\sin x| + C}$$
$$= A|\sin x| \quad (A>0).$$
$$\Rightarrow y = A|\sin x| \text{ or } y = -A|\sin x|$$
$$y(\tfrac{1}{2}\pi) = \tfrac{1}{2}\pi > 0 \Rightarrow y(x) = A|\sin x|$$
$$A|\sin(\tfrac{1}{2}\pi)| = \tfrac{1}{2}\pi \Rightarrow A = \tfrac{1}{2}\pi$$

Thus, the solution is

$$y(x) = \tfrac{1}{2}\pi|\sin x|$$

P25. Solution: $x\frac{dy}{dx} - y = 2x^2y, \; y(1) = 1$

$$x\frac{dy}{dx} = 2x^2y + y$$
$$= y(2x^2+1)$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \underbrace{\frac{2x^2+1}{x}}_{\to 2x + \frac{1}{x}}\,dx$$

$$\ln|y| = x^2 + \ln|x| + C$$
$$|y| = e^{x^2 + \ln|x| + C}$$
$$= A|x|e^{x^2} \quad (A>0)$$
$y = A|x|e^{x^2}$ or $y = -A|x|e^{x^2}$

$y(1) = 1 > 0 \Rightarrow y(x) = A|x|e^{x^2}$
$$1 = A \cdot 1 \cdot e^{1^2}$$
$$\Rightarrow A = \frac{1}{e}$$
Thus, the solution is
$$y(x) = \frac{1}{e}|x|e^{x^2} = |x|e^{x^2-1}$$

P27. Solution: $\frac{dy}{dx} = 6e^{2x-y}$
$$e^y \frac{dy}{dx} = 6e^{2x}$$
$$\int e^y dy = \int 6e^{2x} dx$$
$$e^y = 6 \cdot \left(\frac{e^{2x}}{2}\right) + C$$
$$= 3e^{2x} + C$$
$$y = \ln(3e^{2x} + C)$$
$$y(0) = 0 \Rightarrow 0 = \ln(3e^0 + C)$$
$$\Rightarrow 3 + C = 1 \Rightarrow C = -2$$
Thus, the solution is
$$y(x) = \ln(3e^{2x} - 2)$$

P36. Solution:
$$N(t) = N_0 e^{-kt}.$$
We have $N_0 = 5.0 \times 10^{10}$ and $k = 0.0001216$. We need to solve the time t for that $N(t) = 4.6 \times 10^{10}$, i.e.
$$5.0 \times 10^{10} e^{-0.0001216t} = 4.6 \times 10^{10}$$
$$\Rightarrow e^{-0.0001216t} = 0.92$$
$$t = -\frac{\ln(0.92)}{0.0001216} \approx 685.7 \text{ years}.$$
Thus, this relic is surely faked.

P43. Solution: The environment temperature is $0°C$.
$$\frac{dT}{dt} = k(0 - T) = -kT.$$
$$T(t) = Ae^{-kt}$$
$$T(0) = 25°C \Rightarrow Ae^0 = 25 \Rightarrow A = 25$$
$$\Rightarrow T(t) = 25e^{-kt}$$
Now, use $T(20) = 15$, we get.
$$15 = 25e^{-20k}$$
$$\Rightarrow e^{-20k} = \frac{15}{25} = \frac{3}{5}$$
$$k = \frac{\ln(3/5)}{-20} \approx 0.0255$$
$$\Rightarrow T(t) = 25e^{-0.0255t}$$
Hence, to cool the temperature of the buttermilk to $5°C$, we solve
$$T(t) = 5, \text{ i.e.}$$
$$5 = 25e^{-0.0255t}$$
$$t = \frac{\ln(5/25)}{-0.0255} \approx 63 \text{ mins}$$
It takes about 63 minutes to be at $5°C$.