HW 3 Key

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1) both 0! Cross product gives an orthogonal vector.

2) $P(1,0,-1)$ $Q(5,-2,-1)$ $R(3,0,-2)$

a) $\vec{PQ} = \langle 4,-2,0\rangle$

b) $\vec{PR} = \langle 2,0,-1\rangle$ so $\theta = \cos^{-1}\left(\frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}||\vec{PR}|}\right) = \cos^{-1}\left(\frac{3+0+2}{\sqrt{2}\sqrt{9+4}}\right) = \cos^{-1}\left(\frac{5}{\sqrt{26}}\right)$

c) look at $\vec{PQ}\times\vec{PR} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -2 & 0\\ 2 & 0 & -1\end{vmatrix} = \vec{i}\begin{vmatrix}-2 & 0\\ 0 & -1\end{vmatrix} - \vec{j}\begin{vmatrix}4 & 0\\ 2 & -1\end{vmatrix} + \vec{k}\begin{vmatrix}4 & -2\\ 2 & 0\end{vmatrix} = \langle 2,4,-4\rangle$ this is orthogonal to $\vec{PQ}$ & $\vec{PR}$

to make unit vector: $\frac{\vec{PQ}\times\vec{PR}}{|\vec{PQ}\times\vec{PR}|} = \frac{\langle 2,4,-4\rangle}{\sqrt{4+16+16}} = \frac{\langle 2,4,-4\rangle}{6} = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$ is one. Reverse direction $\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle$ is the other.

d) Area of $\Delta PQR = \frac{1}{2}|\vec{PQ}\times\vec{PR}| = 3$

3) $P(1,4,1)$ $Q(2,3,5)$ $R(1,3,6)$

a) closest z-axis point is $(0,0,5)$, distance is $\sqrt{4+9+0} = \sqrt{13}$

b) $\vec{QP} = \langle -1,1,-4\rangle$

c) $\angle PQR$ we need $\vec{QR} = \langle -1,0,1\rangle$. $\angle PQR = \cos^{-1}\left(\frac{\vec{QP}\cdot\vec{QR}}{|\vec{QP}||\vec{QR}|}\right) = \cos^{-1}\left(\frac{1+0-4}{\sqrt{1+1+16}\sqrt{1+0+1}}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{18}\sqrt{2}}\right) = \cos^{-1}\left(\frac{-3}{6}\right) = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$

d) $\Delta PQR = \frac{1}{2}|\vec{QP}\times\vec{QR}| = \frac{1}{2}\left\|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & -4\\ -1 & 0 & 1\end{vmatrix}\right\| = \frac{1}{2}\left|\vec{i}\begin{vmatrix}1 & -4\\ 0 & 1\end{vmatrix} - \vec{j}\begin{vmatrix}-1 & -4\\ -1 & 1\end{vmatrix} + \vec{k}\begin{vmatrix}-1 & 1\\ -1 & 0\end{vmatrix}\right| = \frac{1}{2}|\langle 1,5,1\rangle| = \frac{1}{2}\sqrt{1+25+1} = \frac{1}{2}\sqrt{27} = \frac{3}{2}\sqrt{3}$

4) $\vec{u} = \langle 7,-2,-4\rangle$ $\vec{v} = \langle 5,1,0\rangle$. cross product is orthogonal

$\vec{u}\times\vec{v} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 7 & -2 & -4\\ 5 & 1 & 0\end{vmatrix} = \vec{i}\begin{vmatrix}-2 & -4\\ 1 & 0\end{vmatrix} - \vec{j}\begin{vmatrix}7 & -4\\ 5 & 0\end{vmatrix} + \vec{k}\begin{vmatrix}7 & -2\\ 5 & 1\end{vmatrix} = \langle 4,20,17\rangle$ make unit vectors with $|\vec{u}\times\vec{v}| = \sqrt{16+400+289} = \sqrt{705}$

so $\langle \frac{4}{\sqrt{705}}, \frac{20}{\sqrt{705}}, \frac{17}{\sqrt{705}}\rangle$ and $\langle \frac{-4}{\sqrt{705}}, \frac{-20}{\sqrt{705}}, \frac{-17}{\sqrt{705}}\rangle$

5) $P(2,1,5)$ $Q(-3,2,5)$ $R(-3,1,6)$

$\vec{PQ} = \langle -5,1,0\rangle$ $\vec{PR} = \langle -5,0,1\rangle$

$|\vec{PQ}\times\vec{PR}| = \left\|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -5 & 1 & 0\\ -5 & 0 & 1\end{vmatrix}\right\| = \left|\vec{i}\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} - \vec{j}\begin{vmatrix}-5 & 0\\ -5 & 1\end{vmatrix} + \vec{k}\begin{vmatrix}-5 & 1\\ -5 & 0\end{vmatrix}\right| = |\langle 0,5,5\rangle| = \sqrt{50} = 5\sqrt{2}$

6) $\vec{u} = \langle 2,3,1\rangle$ $\vec{v} = \langle 0,1,-1\rangle$ $\vec{w} = \langle 0,0,-3\rangle$

$|(\vec{u}\times\vec{v})\cdot\vec{w}| = \begin{vmatrix}2 & 3 & 1\\ 0 & 1 & -1\\ 0 & 0 & -3\end{vmatrix} = 2\begin{vmatrix}1 & -1\\ 0 & -3\end{vmatrix} - 3\begin{vmatrix}0 & -1\\ 0 & -3\end{vmatrix} + 1\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix} = 2(-3) - 0 + 0 = -6$ so 6 really.

7) a) $P(0,3,-2)$ $Q(2,-9,6)$ vector is $\vec{v} = \langle 2,-12,8\rangle$ so

$x = 2t$ $y = 3-12t$ $z = -2-8t$ works

b) $x = -1+3t$ $y = 4-t$ $z = 1-t$ $P(-2,1,3)$ use old vector w/ new points for

$x = -2+3t$ $y = 1-t$ $z = 3-t$

c) $x = 1+2t$ $y = 4-t$ $z = 6+2t$ $P(-2,0,5)$

use the slope vector $\langle 2,-1,2\rangle$ and the point P, any parametrization using a scalar multiple of $\langle 2,-1,2\rangle$ works.

eg) $x = -2+2t$ $y = -t$ $z = 5+2t$ works

$x = -2+4t$ $y = -2t$ $z = 5+4t$ works

or pick a different point on the line eg) $x = 2t$ $y = -1-t$ $z = 7+2t$

8) $3t = s$

$t = 2-s$

$-1+t = -2+s$

$\rangle$ $-1+2t = 0$

$t = \frac{1}{2}$ so $s = \frac{3}{2}$, so $(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2})$ is the intersection.

$2-t$ $\quad$ $2s$ $\quad$ $t = 2-2s$ $\quad$ $t = 4$

$1+t$ $\quad$ $4-s$ $\quad$ $1+2-2s = 4-s$ $\quad$ $-1 = s$

$2+t$ $\quad$ $3-3s$

9) Set them up as a system of equations and show that the system has no solution

10) a) $P(0,2,-1)$ $\vec{n} = \langle 3,-2,-1\rangle$ $3(x-0) + -2(y-2) + -1(z+1) = 0$

$3x - 2y + 4 - z - 1 = 0$

$3x - 2y - z = -3$

b) $P(1,-1,3)$ parallel to $3x+y+z = 7$ so $\vec{n} = \langle 3,1,1\rangle$. So,

$3(x-1) + (y+1) + (z-3) = 0$

$3x - 3 + y + 1 + z - 3 = 0$

$3x + y + z = 5$

c) $P(2,4,5)$ $Q(1,5,7)$ $R(-1,6,8)$ need a normal vector; find it using cross product.

$\vec{PQ} = \langle -1,1,2\rangle$ $\vec{PR} = \langle -3,2,3\rangle$

$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 2\\ -3 & 2 & 3\end{vmatrix} = \vec{i}\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} - \vec{j}\begin{vmatrix}-1 & 2\\ -3 & 3\end{vmatrix} + \vec{k}\begin{vmatrix}-1 & 1\\ -3 & 2\end{vmatrix} = \vec{i}(3-4) - \vec{j}(-3+6) + \vec{k}(-2+3)$

$= \langle -1,-3,1\rangle$

so the plane is $-(x-2) - 3(y-4) + (z-5) = 0$

$-x + 2 - 3y + 12 + z - 5 = 0$

$-x - 3y + z = -9$

11) a & b) $\langle 3,-2,-1\rangle\times\langle 3,1,1\rangle = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -2 & -1\\ 3 & 1 & 1\end{vmatrix} = \vec{i}\begin{vmatrix}-2 & -1\\ 1 & 1\end{vmatrix} - \vec{j}\begin{vmatrix}3 & -1\\ 3 & 1\end{vmatrix} + \vec{k}\begin{vmatrix}3 & -2\\ 3 & 1\end{vmatrix} = \langle -1,-6,9\rangle = \vec{n}$

Setting $y=0$ gives $3x - z = -3$ elimination gives

$3x + z = 5$ $\quad$ $-2z = -8$ so $z = 4$ and $x = \frac{1}{3}$, so $(\frac{1}{3}, 0, 4)$ is a point on both.

So $x = \frac{1}{3} - t$ $y = -6t$ $z = 4 + 9t$ works.

b & c) $\langle 3,1,1\rangle\times\langle -1,-3,1\rangle = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 1 & 1\\ -1 & -3 & 1\end{vmatrix} = \vec{i}\begin{vmatrix}1 & 1\\ -3 & 1\end{vmatrix} - \vec{j}\begin{vmatrix}3 & 1\\ -1 & 1\end{vmatrix} + \vec{k}\begin{vmatrix}3 & 1\\ -1 & -3\end{vmatrix} = \langle 4,-4,-8\rangle = \vec{n}$

setting $y=0$ gives $3x + z = 5$ elimination gives b&c) $\langle 3,1,1\rangle \times \langle -1,-3,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ -1 & -3 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 1 \\ -1 & -3 \end{vmatrix} = \langle 4,-4,-8\rangle = \vec{n}$

setting $y=0$ gives $3x+z=5$, $-x+z=-9$; elimination gives $4x=14$, $x=\frac{7}{2}$ so $\frac{21}{2}+z=\frac{10}{2}$, $z=-\frac{11}{2}$, so $\left(\frac{7}{2}, 0, -\frac{11}{2}\right)$ is a point

$x=\frac{7}{2}+4t \quad y=-4t \quad z=-\frac{11}{2}-8t$

a&c) $\langle 3,-2,-1\rangle \times \langle -1,-3,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ -1 & -3 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & -2 \\ -1 & -3 \end{vmatrix} = \langle -5,-2,-11\rangle$

setting $y=0$ gives $3x-z=-3$, $-x+z=-9$; elim for $2x=-12$, $x=-6$ so $z=-15$ and $(-6,0,-15)$ is a point

$x=-6-5t \quad y=-2t \quad z=-15-11t$

Note: due to the fact that we're parametrizing, there are many equivalent answers to the above 3 problems. If you want to be sure your line is right, make sure its directional vector is a scalar multiple of mine, and that our initial points are on each others' lines.

12) Use $\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$ with normal vectors.

a&b) $\theta=\cos^{-1}\left(\frac{\langle 3,-2,-1\rangle \cdot \langle 3,1,1\rangle}{\sqrt{9+4+1}\sqrt{9+1+1}}\right)=\cos^{-1}\left(\frac{9-2-1}{\sqrt{14}\sqrt{11}}\right)=\cos^{-1}\left(\frac{6}{\sqrt{13}\sqrt{11}}\right)$

b&c) $\theta=\cos^{-1}\left(\frac{\langle 3,1,1\rangle \cdot \langle -1,-3,1\rangle}{\sqrt{11}\sqrt{11}}\right)=\cos^{-1}\left(\frac{-3-3+1}{11}\right)=\cos^{-1}\left(\frac{-5}{11}\right)$

a&c) $\theta=\cos^{-1}\left(\frac{\langle 3,-2,-1\rangle \cdot \langle -1,-3,1\rangle}{\sqrt{14}\sqrt{11}}\right)=\cos^{-1}\left(\frac{-3+6-1}{\sqrt{14}\sqrt{11}}\right)=\cos^{-1}\left(\frac{2}{\sqrt{13}\sqrt{11}}\right)$

13) $S=(0,3,-1)$

a) $d=\frac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$. line is $x=t$, $y=3-2t$, $z=1+t$ so use $P=(0,3,1)$ and $\vec{v}=\langle 1,-2,1\rangle$. $\vec{PS}\times\vec{v}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -2 \\ 1 & -2 & 1 \end{vmatrix}=\vec{i}\begin{vmatrix} 0 & -2 \\ -2 & 1 \end{vmatrix}-\vec{j}\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}+\vec{k}\begin{vmatrix} 0 & 0 \\ 1 & -2 \end{vmatrix}$

$=\langle -4,-2,0\rangle$

so $d=\frac{|\langle -4,-2,0\rangle|}{|\langle 1,-2,1\rangle|}=\frac{\sqrt{16+4}}{\sqrt{1+4+1}}=\frac{\sqrt{20}}{\sqrt{6}}=\sqrt{\frac{10}{3}}$

b) $d=\left|\vec{PS}\cdot\frac{\vec{n}}{|\vec{n}|}\right|$. plane is $3x-2y+4z=1$ so pick any point; say, set $x=y=0$ and find $z=\frac{1}{4}$, so $(0,0,\frac{1}{4})$. Then, $\vec{PS}=\langle 0,3,-5/4\rangle$

$\vec{n}=\langle 3,-2,4\rangle$ so $\frac{\vec{n}}{|\vec{n}|}=\frac{\langle 3,-2,4\rangle}{\sqrt{9+4+16}}=\left\langle \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right\rangle$.

So, $d=\left|\langle 0,3,-\frac{5}{4}\rangle \cdot \left\langle \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right\rangle\right|=\left|0-\frac{6}{\sqrt{29}}-\frac{5}{\sqrt{29}}\right|=\left|-\frac{11}{\sqrt{29}}\right|=\frac{11}{\sqrt{29}}$.