Name:

Math 170: Final Exam

Complete the following problems to the best of your ability. **SHOW ALL OF YOUR WORK.** Unshown work will not be graded. You may use a calculator.

1. I'm a blacksmith who makes enchanted weapons, and I've heard that monsters are coming to attack my town so I want to sell off all my stock. I have 13 iron ingots, 15 magical gems and 12 pieces of wood.

   - A sword takes 3 iron and 1 gem to make.
   - An axe takes 2 iron, 1 gem and 1 wood to make.
   - A staff takes 2 gems and 2 wood to make.

   Represent this scenario as a system of equations, and find out how many of each kind of item I should make in order to use up all of my inventory.

2. Let the matrices A, B and C be as follows:

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 4 & -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 2 & -1 \\ 4 & 5 \end{bmatrix}$$

Find the following new matrices. **Show the steps, calculators are not allowed for this problem except to check your work.**

(a) $A + C^T$

(b) BA

(c) BC

3. Let $A = \{x|$ there is some k so that $x = 2k\}$, and let $B = \{x|0 < x \leq 10\}$. Find the following sets.

(a) $A \cup B$

(b) $A \cap B$

4. Rowena and Carl are playing a game represented by the following payoff matrix:

$$P = \begin{array}{c} \\ r \\ s \\ t \\ u \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & -2 & 3 \\ 2 & 1 & -1 & 6 \\ -1 & -1 & -4 & -2 \\ 1 & 0 & 2 & 3 \end{pmatrix} \end{array}$$

(a) Reduce this game via dominance to a 2x2 game.

(b) Find the optimal mixed strategy for Rowena, the row player.

(c) Find the optimal mixed strategy for Carl, the column player.

(d) What is the expected payoff of this game?

5. The Decision Algorithm Ice Cream Parlour was satisfied with our gelato from the second exam, so they've come to us with another order. We can make a quart of vanilla gelato with 1 egg and 3 cups of cream, and a quart of chocolate with 2 eggs and 3 cups of cream. We have 1000 eggs and 2000 cups of cream in stock. The DAICP is paying more now, and is willing to pay $2 a quart for vanilla and $2.30 a quart for chocolate. How many quarts of each type should we make to maximise our revenue, and what is our maximum revenue?

6. In this problem, a "word" refers to any ordered list of letters (not necessarily one that means anything). For instance, "MATHS" is a word, but so is "SMTHA."

   (a) Suppose we're making random words that are five letters long, and we're allowing repetition of letters. What's the cardinality of the sample space of this experiment?

   (b) If we make a five-letter word at random, calculate the **probabilities** of the following events:

      i. All five letters are vowels.

      ii. The only letters in the word are "O" and "K."

      iii. No letters are used more than once.

      iv. Three of the letters are "F" but neither of the other two are.

7. Suppose a professor (who is not me) is under investigation due to a potentially unfair final exam (again, not me). According to the complaining students, 80% of the students understood the material of the class. However, only 60% of the students that knew the material actually passed. Of the students who didn't know the material, 20% passed through sheer dumb luck. What is the chance that a student who failed the exam knew the material?

8. Suppose a Markov system is represented by the following transition matrix. The time step is one hour.

$$\begin{bmatrix} .2 & .8 \\ .5 & .5 \end{bmatrix}$$

   (a) If we begin the Markov system with a certainty of being in state 2, what is the probability of us ending up in state 1 vs. being in state 2 after 2 full days?

   (b) What is the steady-state vector of this system?

9. Create a truth table for the statement $p \vee (\neg q \wedge p) \rightarrow (\neg p \wedge q) \vee (p \vee q)$

10. Observe the following proof:

    I have been told that if I understand the material from the semester, I will pass the final exam. I did not pass the final exam. Therefore, I must not have understood the material.

    Is this proof via direct reasoning or indirect (contrapositive) reasoning? Explain your answer.

11. Observe the following proof:

    If it's Monday, then I have class. It's not Monday, so I must not have class.

    Does this reasoning hold up? If not, point out the logical fallacy present in the argument and explain why it is false.

12. [Extra Credit] A rational number is one that can be expressed as a fraction in lowest terms. Prove that $\sqrt{2}$ is not rational.