Integration by Partial Fractions

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper ($\deg f(x) < \deg g(x)$)

1. Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \frac{B_3x+C_3}{(x^2+px+q)^3} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are

   (i) One fraction for each power of the irreducible factor that appears

   (ii) The degree of the numerator should be one less than the degree of the denominator

4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.

Example 1 Compute $\displaystyle\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Our first step is to decompose $\dfrac{x+14}{(x+5)(x+2)}$ as

$$\frac{x+14}{(x+5)(x+2)} = \frac{?}{x+5} + \frac{?}{x+2}.$$

There is no indicator of what the numerators should be, so there is work to be done to find them. If we let the numerators be variables, we can use algebra to solve. That is, we want to find constants A and B that make the equation below true for all $x \neq -5, -2$.

$$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}.$$

We solve for A and B by cross multiplying and equating the numerators.

$$\begin{aligned}\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} = \frac{A(x+2)+B(x+5)}{(x+5)(x+2)} \Longrightarrow x+14 &= A(x+2)+B(x+5)\\ &= Ax+2A+Bx+5B\\ &= (A+B)x+2A+5B\end{aligned}$$

This leaves us with the system of equations

$$A + B = 1 \qquad\qquad 2A + 5B = 14.$$

Rearranging the first we obtain $B = 1 - A$. Substituting this into the second gives

$$14 = 2A + 5B = 2A + 5\left(1 - A\right) = 2A + 5 - 5A = 5 - 3A \Longrightarrow 9 = -3A \Longrightarrow \boxed{A = -3} \Longrightarrow \boxed{B = 4}$$

So,

$$\int \frac{x + 14}{(x + 5)\,(x + 2)}\, dx = \int \frac{-3}{x + 5} + \frac{4}{x + 2}\, dx = \boxed{-3 \ln|x + 5| + 4 \ln|x + 2| + C}$$

This method isn't a new way to integrate. This method is just an exercise in algebraic manipulation to rearrange a seemingly complicated integral to turn it into an integral that can be done using the methods we are familiar with. Let's now see an example of when there is a repeated irreducible factor on the denominator.

Example 2 Find $\displaystyle\int \frac{5x - 2}{(x + 3)^2}\, dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x - 2}{(x + 3)^2} = \frac{5x - 2}{(x + 3)(x + 3)} \neq \frac{A}{x + 3} + \frac{B}{x + 3} = \frac{A + B}{x + 3}.$$

However, it can be expressed in the form:

$$\frac{5x - 2}{(x + 3)^2} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2}.$$

$$\frac{5x - 2}{(x + 3)^2} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} = \frac{A(x + 3) + B}{(x + 3)^2} \Longrightarrow 5x - 2 = A(x + 3) + B$$

$$\begin{aligned} x = -3: \quad & 5(-3) - 2 = A\left((-3) + 3\right) + B \\ \Longrightarrow \quad & -17 = 0 + B \\ \Longrightarrow \quad & -17 = B \end{aligned}$$

$$\begin{aligned} 5x - 2 &= A(x + 3) - 17 \\ &= Ax + 3A - 17 \\ \Longrightarrow \quad 5x &= Ax \\ \Longrightarrow \quad 5 &= A \end{aligned}$$

$$\int \frac{5x - 2}{(x + 3)^2}\, dx = \int \frac{5}{x + 3} - \frac{17}{(x + 3)^2}\, dx = \boxed{5 \ln|x + 3| + \frac{17}{x + 3} + C}$$

The final thing we should look at is the case when there is an irreducible polynomial of degree higher than 1 on the denominator of the rational expression.

Example 3 Compute $\int \frac{-2x+4}{(x^2+1)(x-1)}\,dx$.

The process follows as before. The most common mistake here is to not choose the right numerator for the term with the x^2+1 on the denominator. The term of the numerator should have degree 1 less than the denominator - so this term should have an $Ax+B$ on the numerator. Let's see:

$$\frac{-2x+4}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} = \frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)}$$
$$\implies -2x+4 = (Ax+B)(x-1)+C(x^2+1).$$

$$\begin{aligned} x=1: \quad & -2(1)+4 = (Ax+B)(1-1)+C((1)^2+1) \\ \implies \quad & 2 = 0+2C \\ \implies \quad & 1 = C \end{aligned}$$

So now we have,

$$\begin{aligned} & -2x+4 = (Ax+B)(x-1)+(x^2+1) \\ \implies \quad & -x^2-2x+3 = (Ax+B)(x-1) \\ \implies \quad & -(x-1)(x+3) = (Ax+B)(x-1) \\ \implies \quad & -(x+3) = Ax+B \\ \implies \quad & -1 = A \\ & -3 = B. \end{aligned}$$

Thus our integral becomes,

$$\begin{aligned} \int \frac{-2x+4}{(x^2+1)(x-1)}\,dx &= \int \frac{-x-3}{x^2+1} + \frac{1}{x-1}\,dx \\ &= -\int \frac{x}{x^2+1}\,dx - 3\int \frac{1}{x^2+1}\,dx + \int \frac{1}{x-1}\,dx \\ &= \boxed{-\frac{1}{2}\ln(x^2+1) - 3\tan^{-1}(x) + \ln|x-1| + C} \end{aligned}$$

Where the first two integrals are solved with a u-substitution and trigonometric substitution, respectively.

Practice Problems

Try some of the problems below. If you get stuck, don't worry! There are hints on the next page! But do try without looking at them first, chances are you won't get hints on your exam.

1. $\int \frac{x-9}{(x+5)(x-2)}\,dx$
2. $\int \frac{1}{(t+4)(t-1)}\,dt$
3. $\int_2^3 \frac{1}{x^2-1}\,dx$
4. $\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$
5. $\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$
6. $\int \frac{x^2+2x-1}{x^3-x}\,dx$

7. $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$

8. $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$

9. $\displaystyle\int \frac{1}{s^2(s-1)^2}\,ds$

10. $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

11. $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,dx$

12. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

Challenge Problems

Below are some harder problems that require a little more thinking/algebraic manipulation to make the substitutions work.

1. $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

2. $\displaystyle\int \frac{x^3-4x-10}{x^2-x-6}\,dx$

3. $\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx$

4. $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$

5. $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$

6. $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

Answers to Practice Problems

1. $2\ln|x+5| - \ln|x-2| + C$

2. $\dfrac{1}{5}\left(\ln|t-1| - \ln|t+4|\right) + C$

3. $\dfrac{1}{2}\ln\left(\dfrac{3}{2}\right)$

4. $\ln\left(\dfrac{27}{32}\right)$

5. $\dfrac{9}{5}\ln\left(\dfrac{8}{3}\right)$

6. $\ln|x| + \ln|x-1| - \ln|x+1| + C$

7. $\dfrac{1}{36}\left(\dfrac{6}{x+5} + \ln|x-1| - \ln|x+5|\right) + C$

8. $\dfrac{1}{2}\left(3\ln|2x+1| - 2\ln|x-2| - \dfrac{4}{x-2}\right) + C$

9. $2\ln|s| - \dfrac{1}{s} - 2\ln|s-1| - \dfrac{1}{s-1} + C$

10. $2\ln|x| + \dfrac{1}{x} + 3\ln|x+2| + C$

11. $\dfrac{1}{2}\left(4\ln|x| - \ln(x^2+3)\right) + C$

12. $\dfrac{1}{6}\left(6\ln|x-1| - 3\ln(x^2+9) - 2\tan^{-1}\left(\dfrac{x}{3}\right)\right) + C$

Answers to Challenge Problems

1. $\dfrac{1}{6}\left(7 + 6\ln\left(\dfrac{2}{3}\right)\right)$

2. $\dfrac{1}{2}\left(x^2 + 2x + 2\ln|x-3| + 4\ln|x+2|\right) + C$

3. $\dfrac{1}{2}\left(\ln(x^2+1) + \sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{2}x}{2}\right)\right) + C$

4. $\dfrac{1}{2}\left(2\ln|x-1| + \dfrac{2}{x-1} - \ln(x^2+1) + 2\tan^{-1}(x)\right) + C$

5. $\dfrac{1}{2}\left(\ln(x^2+2x+5) + 3\tan^{-1}\left(\dfrac{x+1}{2}\right)\right) + C$

6. $\dfrac{1}{2}\left(2\tan^{-1}(x) - \dfrac{1}{x^2+1}\right) + C$