Trigonometric Substitution

Common Trig Substitutions: The following is a summary of when to use each trig substitution.

Integral contains:	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\left[0, \frac{\pi}{2}\right)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

If you are worried about remembering the identities, then don't! They can all be derived easily, assuming you know three basic ones (which by now you should):

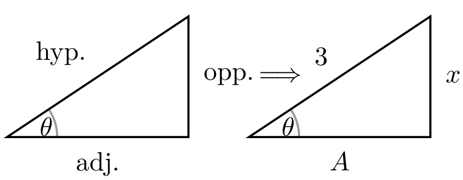
$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Example 1 Find

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$$

$$x = 3\sin(\theta)$$
$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$dx = 3\cos(\theta)\, d\theta$$

$$x = 3\sin(\theta) \implies \frac{x}{3} = \sin(\theta)$$



$$A^2 + x^2 = 3^2$$
$$A^2 = 3^2 - x^2$$
$$A = \sqrt{3^2 - x^2}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{3^2 - x^2}}{x}$$

$$\begin{aligned}
\int \frac{\sqrt{9 - x^2}}{x^2}\, dx &= \int \frac{\sqrt{3^2 - 3^2\sin^2(\theta)}}{3^2\sin^2(\theta)} \cdot 3\cos(\theta)\, d\theta \\
&= \int \frac{\cancel{3}\sqrt{1 - \sin^2(\theta)}}{\cancel{3}^{\cancel{2}}\sin^2(\theta)} \cdot \cancel{3}\cos(\theta)\, d\theta \\
&= \int \frac{\sqrt{\cos^2(\theta)}}{\sin^2(\theta)} \cdot \cos(\theta)\, d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\, d\theta \\
&= \int \cot^2(\theta)\, d\theta \\
&= \int \csc^2(\theta) - 1\, d\theta \\
&= -\cot(\theta) - \theta + C \\
&= \boxed{-\frac{\sqrt{3^2 - x^2}}{x} - \arcsin(\theta) + C}
\end{aligned}$$

This is a common process in trig substitution. When you substitute back for your original variable, in this case x, you will always be able to find the correct substitutions by drawing out and labelling a right triangle correctly.

Example 2 Evaluate

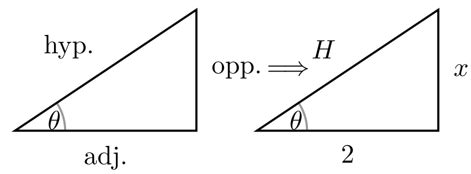
$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

$$x = 2\tan(\theta)$$
$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$dx = 2\sec^2(\theta)\ d\theta$$

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{2\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{2^2\tan^2(\theta)+2^2}}\,d\theta \\
&= \int \frac{\cancel{2}\sec^2(\theta)}{2^2\tan^2(\theta)\,\cancel{2}\sqrt{\tan^2(\theta)+1}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec(\theta)}{2^2\tan^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{u^2}\,du \\
&= -\frac{1}{4}\frac{1}{u} + C \\
&= -\frac{1}{4\sin(\theta)} + C \\
&= -\frac{1}{4}\csc(\theta) + C \\
&= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}
\end{aligned}$$

$$u = \sin(\theta)$$
$$du = \cos(\theta)$$

$$x = 2\tan(\theta) \Longrightarrow \frac{x}{2} = \tan(\theta)$$



$$H^2 = x^2 + 2^2$$
$$H = \sqrt{x^2+4}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hyp.}}{\text{opp.}} = \frac{\sqrt{x^2+4}}{x}$$

The main thing you need to know when doing these types of integrals is **don't be put off by all the lines of algebra!** Without actually reading them, these past two examples seem scary - look at how many lines of algebra there are! But if you look at each step, the algebra is very straightforward. There are no tricks, you can do all of them just by following your nose. The more problems you try, the more you'll realise that they are all the same!

Practice Problems

Try some of the problems below. If you get stuck, don't worry! There are hints on the next page! But do try without looking at them first, chances are you won't get hints on your exam.

1. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$
2. $\displaystyle\int x^3\sqrt{9-x^2}\,dx$
3. $\displaystyle\int \frac{x^3}{\sqrt{x^2-9}}\,dx$
4. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$
5. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,dx$
6. $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$
7. $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$
8. $\displaystyle\int \frac{x^3}{\sqrt{x^2+100}}\,dx$
9. $\displaystyle\int \frac{1}{\sqrt{x^2+16}}\,dx$
10. $\displaystyle\int \frac{x^5}{\sqrt{x^2+2}}\,dx$
11. $\displaystyle\int \sqrt{1-4x^2}\,dx$
12. $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx$
13. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$
14. $\displaystyle\int \frac{1}{x\sqrt{5-x^2}}\,dx$
15. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{1}{x^5\sqrt{9x^2-1}}\,dx$
16. $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$
17. $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$
18. $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$

Challenge Problems

Below are some harder problems that require a little more thinking/algebraic manipulation to make the substitutions work.

1. $\displaystyle\int \sqrt{5+4x-x^2}\,dx$
2. $\displaystyle\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$
3. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$
4. $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$
5. $\displaystyle\int \sqrt{x^2+2x}\,dx$
6. $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

Hints to Practice Problems

1. $x = 3\sec(\theta)$
2. $x = 3\sin(\theta)$
3. $x = 3\tan(\theta)$
4. $x = 4\sin(\theta)$
5. $x = \sec(\theta)$
6. $x = \sec(\theta)$
7. $x = 5\sin(\theta)$
8. $x = 10\tan(\theta)$
9. $x = 4\tan(\theta)$
10. $x = \sqrt{2}\tan(\theta)$
11. $2x = \sin(\theta)$
12. u-sub.
13. $x = 3\sec(\theta)$
14. $x = \sqrt{5}\sin(\theta)$
15. $3x = \sec(\theta)$
16. u-sub.
17. $x = \tan(\theta)$
18. $x = 5\sin(x)$

Hints to Challenge Problems

All of these problems require a complete the square step first. Once you do this, these are the substitutions.

1. $x - 2 = 3\sin(\theta)$
2. $x - 3 = 2\tan(\theta)$
3. $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan(\theta)$
4. $2x - 1 = 2\sin(\theta)$
5. $x + 1 = \sec(\theta)$
6. $x - 1 = \tan(\theta)$