Trigonometric Integrals

Trigonometric Integrals involve, unsurprisingly, the six basic trigonometric functions you are familiar with

$$\cos(x), \quad \sin(x), \quad \tan(x), \quad \sec(x), \quad \csc(x), \quad \cot(x).$$

The general idea is to use trigonometric identities to transform seemingly difficult integrals into ones that are more manageable - often the integral you take will involve some sort of u-substitution to evaluate. Let's remind ourselves of the main trig identities that are useful to us.

$$\cos^2(x) + \sin^2(x) = 1, \quad \sin(2x) = 2\cos(x)\sin(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x).$$

Particularly for trigonometric integrals, the third identity is most helpful if we rearrange and obtain the following:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}, \quad \sin^2(x) = \frac{1-\cos(2x)}{2x}.$$

If you are unfamiliar with this you should rearrange the final identity to see if you can obtain these two.

To tackle these trigonometric integrals, we usually decide how to proceed based on what the powers of the trig functions in the integrand have. Namely, we have the following three cases: For a general integral

$$\int \sin^m(x)\cos^n(x)\,dx,$$

Case 1: If m **is odd** we can write $m = 2k+1$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to obtain:

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2k}(x)\cos^n(x)\sin(x)\,dx = \int \left(1-\cos^2(x)\right)^k \cos^n(x)\sin(x)\,dx = -\int (1-u^2)^k u^n\,du,$$

where the final integral on the right is obtained by a u-substitution with $u = \cos(x)$.

Case 2: If m **is even and** n **is odd** we can write $n = 2k+1$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to obtain:

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\cos^{2k}(x)\cos(x)\,dx = \int \sin^m(x)\left(1-\sin^2(x)\right)^k \cos(x)\,dx = \int u^m(1-u^2)^k\,du,$$

where the final integral on the right is obtained by a u-substitution with $u = \sin(x)$.

Case 3: If **both** m **and** n **are even** we can write $m = 2k$ and $n = 2j$ and use the identities

$$\cos^2(x) = \frac{1+\cos(2x)}{2}, \text{ and } \sin^2(x) = \frac{1-\cos(2x)}{2x},$$

to obtain:

$$\int \sin^m(x)\cos^n(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)^k \left(\frac{1+\cos(2x)}{2}\right)^j\,dx$$

and reduce it to an integral in terms of lower powers of $\cos(2x)$.

So that's all the theory - let's now use it in some examples.

Example 1 Find

$$\int \sin^3(x)\,dx.$$

Here we have an odd power of $\sin(x)$, so we are in case 1. The idea then is that we want to *peel away* one of the $sin(x)$ terms and then use the identity $\sin^2(x) = 1 - \cos^2(x)$ on the ones that are left. So,

$$\int \sin^3(x)\,dx = \int \sin^2(x)\sin(x)\,dx = \int (1-\cos^2(x))\sin(x)\,dx.$$

We are now left with some expression of $\cos(x)'s$ and a single $\sin(x)$ term - it is this lone $\sin(x)$ term that is going to allow us to successfully u-sub.. So, letting $u = \cos(x)$ we have.

$$\int (1-\cos^2(x))\sin(x)\,dx = -\int (1-u^2)\,du = -u + \frac{u^3}{3} + C = \boxed{-\cos(x) + \frac{\cos^3(x)}{3} + C}$$

This is how all integrals that fall into case 1 are going to go. The same can be said for case 2 - the only difference being we make a u-sub. of $u = \sin(x)$ instead. The below example is one where the $(1-u^2)$ term has a higher power than just 1. It is reasonable to expect that a $(1-u^2)^2$ or $(1-u^2)^3$ term appears, but higher than this really doesn't test anything more so you shouldn't worry about that. Let's see how it goes.

Example 2 Evaluate

$$\int \sin^2(x)\cos^5(x)\,dx = \int \sin^2(x)\cos^4(x)\cos(x)\,dx = \int \sin^2(x)(1-\sin^2(x))^2\cos(x)\,dx,$$

letting $u = \sin(x)$, so $du = \cos(x)\,dx$, we get

$$\begin{aligned}\int \sin^2(x)(1-\sin^2(x))^2\cos(x)\,dx &= \int u^2(1-u^2)^2\,du = \int u^2(1-2u^2+u^4)\,du = \int u^2 - 2u^4 + u^6\,du \\ &= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C = \boxed{\frac{\cos^3(x)}{3} - \frac{2\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C}\end{aligned}$$

Finally let's explore case 3, when the powers of $\sin(x)$ and $\cos(x)$ are both even.

Example 3 Find

$$\int \sin^2(x)\cos^2(x)\,dx.$$

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}(1-\cos(2x)) \cdot \frac{1}{2}(1+\cos(2x))\,dx = \frac{1}{4}\int 1-\cos^2(x)\,dx \\ &= \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)\,dx = \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}(1+\cos(4x))\,dx \\ &= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos(4x)\,dx = \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}\end{aligned}$$

Here we made use of the identities $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$.

What about the other trig functions?

So far we have only seen examples of trigonometric integrals involving $\sin(x)$ and $\cos(x)$. The strategy if other pairs of trig. functions show up is similar to what we have done before. By manipulating the identity $\cos^2(x)+\sin^2(x)=1$ we obtain the identities

$$1+\tan^2(x)=\sec^2(x) \text{ and } \cot^2(x)+1=\csc^2(x),$$

which can be used to make the required u-substitution - by again *peeling away* the correct combination of trig. functions to account for the du. You should spend a few minutes analysing the next couple of examples to convince yourself that if you are comfortable with the first three examples, these next ones are not so surprising.

Example 4 Evaluate

$$\int \tan^6(x)\sec^4(x)\,dx.$$

$$\int \tan^6(x)\sec^4(x)\,dx = \int \tan^6(x)\sec^2(x)\sec^2(x)\,dx = \int \tan^6(x)(1+\tan^2(x))\sec^2(x)\,dx.$$

Letting $u=\tan(x)$, so $du=\sec^2(x)\,dx$, we obtain

$$\int \tan^6(x)(1+\tan^2(x))\sec^2(x)\,dx = \int u^6(1+u^2)\,du = \int u^6+u^8\,du = \frac{u^7}{7}+\frac{u^9}{9}+C = \boxed{\frac{\tan^7(x)}{7}+\frac{\tan^9(x)}{9}+C}$$

Example 5 Find

$$\int \tan^5(x)\sec^7(x)\,dx.$$

$$\int \tan^5(x)\sec^7(x)\,dx = \int \tan^4(x)\sec^6(x)\tan(x)\sec(x)\,dx = \int (\sec^2(x)-1)^2\sec^6(x)\tan(x)\sec(x)\,dx.$$

Letting $u=\sec(x)$, so $du=\tan(x)\sec(x)\,dx$, we obtain

$$\begin{aligned}\int (\sec^2(x)-1)^2\sec^6(x)\tan(x)\sec(x)\,dx &= \int (u^2-1)^2u^6\,du = \int u^{10}-2u^8+u^6\,du = \frac{u^{11}}{11}-\frac{2u^9}{9}+\frac{u^7}{7}+C\\ &= \boxed{\frac{\sec^{11}(x)}{11}-\frac{2\sec^9(x)}{9}+\frac{\sec^7(x)}{7}+C}\end{aligned}$$

Example 6 Evaluate

$$\int \csc^4(x)\cot^6(x)\,dx.$$

$$\int \csc^4(x)\cot^6(x)\,dx = \int \csc^2(x)\cot^6(x)\csc^2(x)\,dx = \int (\cot^2(x)+1)\cot^6(x)\csc^2(x)\,dx.$$

Letting $u=\cot(x)$, so $du=-\csc^2(x)\,dx$, we obtain

$$\int (\cot^2(x)+1)\cot^6(x)\csc^2(x)\,dx = -\int (u^2+1)^2u^6\,du = -\int u^8+u^6\,du = -\frac{u^9}{9}-\frac{u^7}{7}+C = \boxed{-\frac{\cot^9(x)}{9}-\frac{\cot^7(x)}{7}+C}$$

Of course this last example can be done with just a u-sub. if you simplify the integrand and write it in terms of $\sin(x)$ and $\cos(x)$. Which you can often do in these cases.

Practice Problems

Try some of the problems below. If you get stuck, don't worry! Hints are given below! But do try without looking at them first, chances are you won't get hints on your exam.

1. $\int \sin^3(x)\,dx$
2. $\int \sin^6(x)\cos^3(x)\,dx$
3. $\int_{\pi/2}^{3\pi/4} \sin^5(x)\cos^3(x)\,dx$
4. $\int_0^{\pi/2} \cos^5(x)\,dx$
5. $\int_0^{\pi/2} \cos^2(x)\,dx$
6. $\int \sin^2(2x)\,dx$
7. $\int \cos^2(x)\tan^3(x)\,dx$
8. $\int \cot^5(x)\sin^4(x)\,dx$
9. $\int \sec^2(x)\tan(x)\,dx$
10. $\int \sec^4(x)\,dx$
11. $\int \sec^6(x)\,dx$
12. $\int_0^{\pi/4} \sec^4(x)\tan^4(x)\,dx$
13. $\int_0^{\pi/3} \tan^5(x)\sec^4(x)\,dx$
14. $\int \tan^3(x)\sec^5(x)\,dx$
15. $\int \tan^3(x)\sec(x)\,dx$
16. $\int \tan^3(x)\sec(x)\,dx$
17. $\int_{\pi/6}^{\pi/2} \cot^2(x)\,dx$
18. $\int \tan^2(x)\,dx$

Challenge Problems

1. $\int \cot^5(x)\sin^4(x)\,dx$
2. $\int_0^{\pi} \sin^2(x)\cos^4(x)\,dx$
3. $\int \sin^2(\pi x)\cos^5(\pi x)\,dx$
4. $\int \tan^5(x)\,dx$
5. $\int_{\pi/4}^{\pi/2} \cot^3(x)\,dx$
6. $\int x\sec(x)\tan(x)\,dx$

Hints to Practice Problems

1. Case 1
2. Case 2
3. Case 1
4. Case 2
5. Case 3
6. Case 3
7. Case 1
8. Case 1
9. Straight in with u-sub.
10. Example 4
11. Example 4
12. Example 4
13. Example 4
14. Example 5
15. Example 4
16. Example 5
17. $\cot^2(x) = \csc^2(x) - 1$
18. $\tan^2(x) = \sec^2(x) - 1$

Hints to Challenge Problems

1. Simplify and $\cos^2(x) = 1 - \sin^2(x)$
2. $2\sin(x)\cos(x) = \sin(2x)$
   $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
3. Case 2, don't be afraid of the π
4. $\tan^2(x) = \sec^2(x) - 1$ and expand
5. $\cot^2(x) = \csc^2(x) - 1$
6. Parts with $g(x) = x$