The Taylor Remainder

Taylor's Formula: If $f(x)$ has derivatives of all orders in a n open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Definitions: The second equation is called **Taylor's formula**. The function $R_n(x)$ is called the remainder of order n or the error term for the approximation of $f(x)$ by $P_n(x)$ over I.

If $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in I$, we say that the Taylor Series generated by $f(x)$ at $x = a$ **converges** to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Often we can estimate $R_n(x)$ without knowing the value of c.

The Remainder Estimation Theorem: If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Example 1: Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

$f(x)$ has derivatives of all orders on $(-\infty, \infty)$. Using the Taylor Polynomial generated by $f(x) = e^x$ at $a = 0$ and Taylor's formula, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

where $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1}$ for some c between 0 and x. Recall that e^x is an increasing function, so if $0 < |c| < |x|$, we know $1 < e^{|c|} < e^{|x|}$. Thus,

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^c|x|^{n+1}}{(n+1)!} \leq \lim_{n\to\infty} \frac{e^{|x|}|x|^{n+1}}{(n+1)!} = e^{|x|} \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Hence, since $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, the Taylor series converges to e^x on $(-\infty, \infty)$.

Example 2: Estimate the error if $P_2(x) = 1 - \dfrac{x^2}{2}$ is used to estimate the value of $\cos(x)$ at $x = 0.6$.

We are estimating $f(x) = \cos(x)$ with its 2^{nd} degree Taylor polynomial (centred at zero), so we can bound the error by using the remainder estimation Theorem, with $n = 2$. So,

$$\text{Error} = |R_2(x)|\Big|_{x=0.6} = \left|\frac{f^3(c)}{3!}x^3\right|\Big|_{x=0.6} = \frac{|\sin(c)|}{3!}|x|^3\Big|_{x=0.6} \leq \frac{1}{3!}|x|^3\Big|_{x=0.6} = 0.036.$$

Example 3: For approximately what values of x can you replace $\sin(x)$ by $x - \dfrac{x^3}{6}$ with an error of magnitude no greater than 4×10^{-3}?

We wish to estimate $f(x) = \sin(x)$ with its 3^{rd} degree Taylor polynomial (centred at zero), so first we bound the error using the remainder estimation theorem:

$$\text{Error} = |R_3(x)| = \left|\frac{f^4(c)}{4!}x^4\right| = \frac{|\sin(c)|}{4!}x^4 \leq \frac{1}{4!}x^4.$$

We want the error to be less than or equal to 4×10^{-3}, so we solve the following inequality,

$$\frac{1}{4!}x^4 \leq 0.004 \Longrightarrow |x| \leq \sqrt[4]{4! \cdot 0.004} \approx 0.556.$$

Thus the values of x in the interval $[-0.556, 0.556]$ can be approximated to the desired accuracy.

Note that the approximations in the previous two examples can be improved by using the Alternating Series Estimation Theorem instead.

Example 4: Use the remainder estimation theorem to estimate the maximum error when approximating $f(x) = e^x$ by $P_2(x) = 1 + x + \dfrac{x^2}{2}$ on the interval $\left[-\dfrac{5}{6}, \dfrac{5}{6}\right]$.

We wish to estimate $f(x) = e^x$ with its 2^{nd} degree Taylor polynomial (centred at zero), so first lets bound the error for a general x:

$$\text{Error} = |R_2(x)| = \left|\frac{f^{(3)}(c)}{3!}x^3\right| \leq \frac{e^c}{3!}|x|^3,$$

where c lies between $a = 0$ and x. Now, since we are looking at only the interval $\left[-\dfrac{5}{6}, \dfrac{5}{6}\right]$, we have that $|c| < \dfrac{5}{6}$ for each x in this interval. So, $e^c \leq e^{5/6}$, since e^x is an increasing function.
Now we apply some guessing work. We are approximating values of e^x, so it doesn't seem right to use one of these values in our bound (if we could get the value of $e^{5/6}$ then why would we merely approximate?), so we should bound $e^{5/6}$. There are many ways to do this, and you may use any justification you see fit. We shall use,

$$e^{5/6} < e^1 < 3.$$

Thus, for $|x| \leq \dfrac{5}{6}$, the error can be bounded by

$$\text{Error} \leq \frac{e^c}{3!}|x|^3 \leq \frac{e^{5/6}}{3!}|x|^3 \leq \frac{3}{3!}\left|\frac{5}{6}\right|^3 = 0.289.$$

Practice Problems

Estimate the maximum error when approximating the following functions with the indicated Taylor polynomial centred at a, on the given interval.

1. $f(x) = \sqrt{x}$,
   $n = 2$, $a = 4$,
   $4 \leq x \leq 4.2$
2. $f(x) = x^{-2}$,
   $n = 2$, $a = 1$,
   $0.9 \leq x \leq 1.1$
3. $f(x) = x^{2/3}$,
   $n = 3$, $a = 1$,
   $0.8 \leq x \leq 1.2$
4. $f(x) = \sin(x)$,
   $n = 4$, $a = \pi/6$,
   $0 \leq x \leq \pi/3$
5. $f(x) = \sec(x)$,
   $n = 2$, $a = 0$,
   $-0.2 \leq x \leq 0.2$
6. $f(x) = \ln(1 + 2x)$,
   $n = 2$, $a = 0$,
   $0.5 \leq x \leq 1.5$
7. $f(x) = e^{x^2}$,
   $n = 3$, $a = 0$,
   $0 \leq x \leq 0.1$
8. $f(x) = x\ln(x)$,
   $n = 3$, $a = 1$,
   $0.5 \leq x \leq 1.5$
9. $f(x) = x\sin(x)$,
   $n = 4$, $a = 0$,
   $-1 \leq x \leq 1$

Answers to Practice Problems

1. $\dfrac{0.008}{512}$
2. $\dfrac{0.004}{0.59049}$
3. $\dfrac{56 \cdot 0.0016}{1944 \cdot (0.8)^{10/3}}$
4. $\dfrac{1}{120}\left(\dfrac{\pi}{6}\right)^5$
5. ≈ 1.085
6. $\dfrac{1}{64}$
7. $\dfrac{e^{0.01} \cdot 12.4816}{24} \cdot (0.1)^4$
8. $\dfrac{1}{24}$
9. $\dfrac{1}{24}$