MATH 172 - Mathematical Modelling for the Life Sciences - Spring 20
Single Species Population Modelling - The Logistic Model with Allee Effect

Allees well that ends well

We have already talked about how populations grows in terms of the logistic model and how the carrying capacity works – it governs the behaviour of populations with high densities. But we have not yet considered how a small population might affect growth.

Our initial assumptions say that the growth rate of a population will decrease at higher densities and increase at lower densities due to competition for limited resources – food and land for example. But if you had a population consisting of, say, a single tiger, then it does not matter how much food or land that tiger has, the population will die out because it has nothing to breed with. **The Allee effect**, named after Walter Clyde Allee, is the principle that individuals within a population require the presence of other individuals in order to survive and reproduce successfully. Thus when the population size is too small, it will not be able to maintain a positive growth rate. **The logistic equation with Allee effect** has the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{A} - 1\right),$$

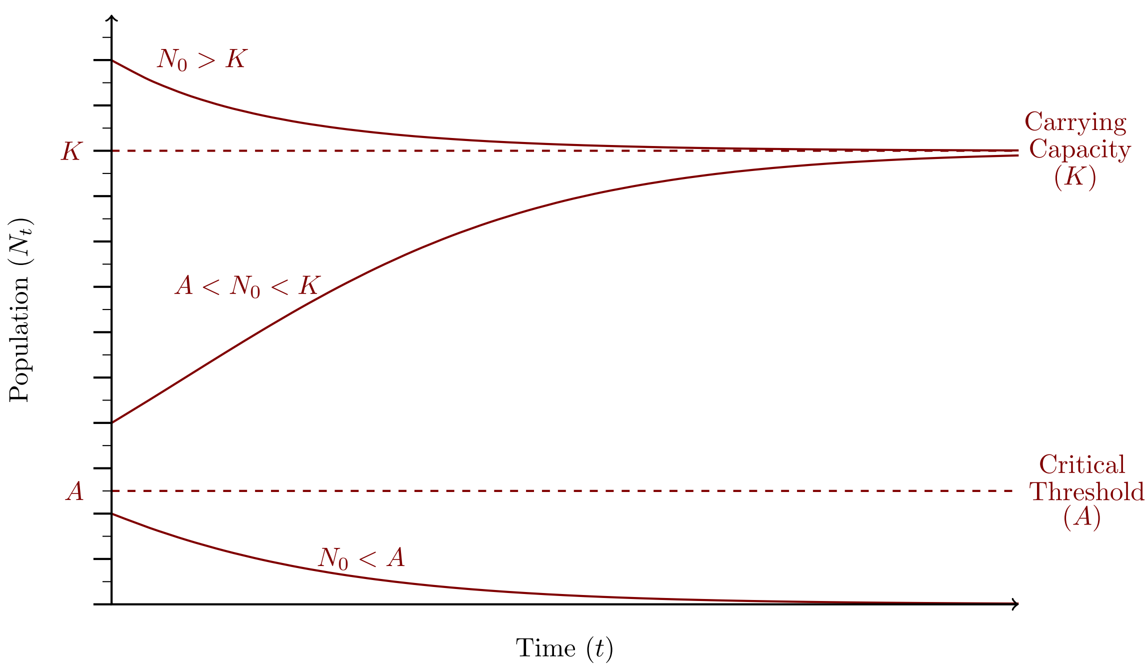
where, as before, r is the intrinsic growth rate, K is the carrying capacity and the new variable, A, is the minimal size of the population required to survive – called the **Allee threshold**.

We will assume that $A < K$, since we want A to represent the *minimum* size the population needs and K should be the *maximum* size it can sustain. We will see that the long term outcome of a population modelled by this equation depends on whether the initial value is above or below the value of A.

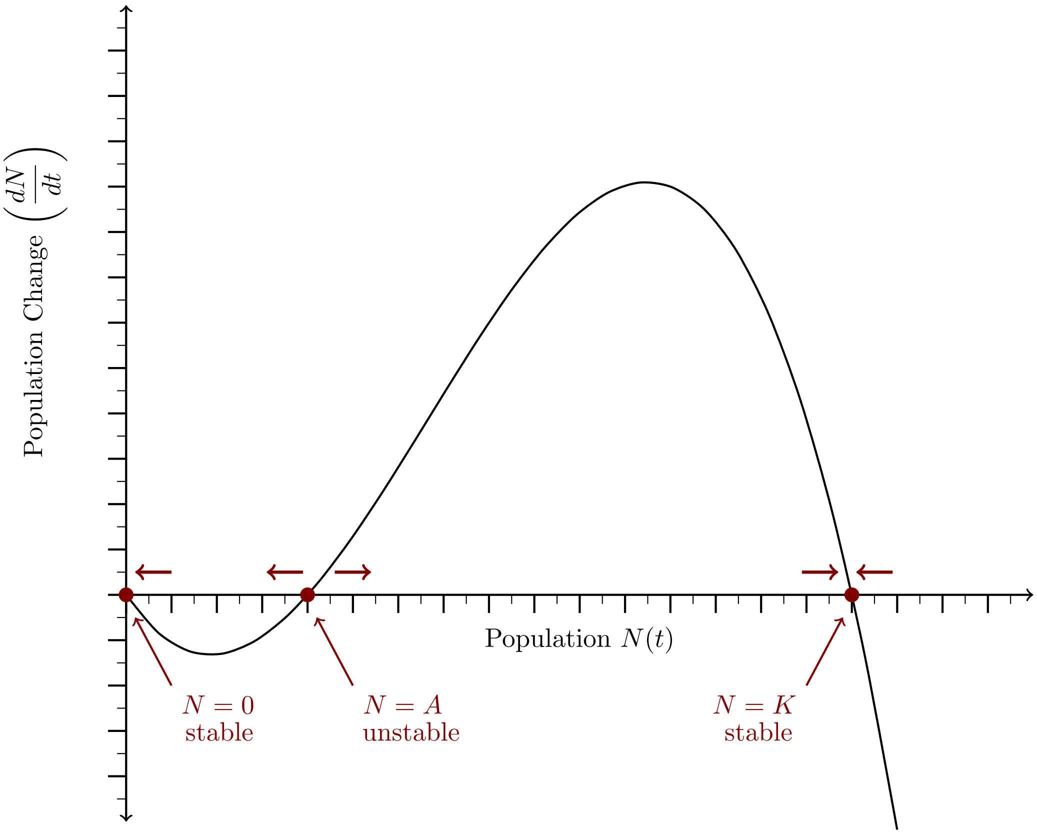
Example 1: Write a possible differential equation for a population whose growth is modelled by a logistic equation with Allee effect if the intrinsic growth rate is 15% the carrying capacity is 800 individuals and at least 100 individuals are required in order for the population to survive.

$$\frac{dN}{dt} = 0.15N\left(1 - \frac{N}{800}\right)\left(\frac{N}{100} - 1\right)$$

As with the logsitic model, if $N > K$ then $dN/dt < 0$ and the population is decreasing. If $A < N < K$, then $dN/dt > 0$ and the population is increasing. The new behaviour of this model is when $N < A$, which given that $A < K$ we can see that $dN/dt < 0$ and so the population decreases.



With the addition of this new term, the model picks up another equilibrium point at $N = A$, bringing the total to three – the points $N = 0$ and $N = K$ survive from the previous logistic model. Noticeably, the point $N = 0$ now becomes a stable equilibrium point, since any population level below A will result in distinction. The point $N = A$ is unstable, with the population either growing to its maximum capacity or dying out and the points $N = K$ remains a stable equilibrium point.



Symbol	Meaning
A	Allee threshold
b	Instantaneous birth rate
B	Number of births
d	Instantaneous death rate
D	Number of deaths
ΔN	Change in population size between time t and $t+1$
$\frac{dN}{dt}$	Population growth rate
e	Euler's number
E	Number of emigrants leaving the population
I	Number of immigrants entering the population
K	Carrying capacity
λ	Finite rate of increase
N	Population size
N_0	Initial population
N_t	Population size at time t
r	Instantaneous rate of increase
r_d	Discrete growth factor
t	Time
t_D	Doubling time