MATH 172 - Mathematical Modelling for the Life Sciences - Spring 20

Single Species Population Modelling - The Logistic Model

Nothing lasts forever

Let's address the biggest issue with our current exponential model, constant birth and death rates. More specifically the issue is what this implies - that resources such as food and space are infinite, and a population can grow without bound. Intuitively, the higher the population, the more resources are being used. So, as resources deplete, the birth rate should slow down. If it doesn't, the resources run dry and everybody dies. We start with the familiar exponential model

$$\frac{dN}{dt} = (b' - d')N, \tag{1}$$

but instead of b' and d' being constant, we will modify them to be density dependent and reflect crowding.

As a population becomes more crowded, we expect the per capita birth rate to decrease, since there are fewer resources per organism available. So let's start with the simplest decreasing function - a straight line with negative slope.

$$b' = b - aN \tag{2}$$

where b and a are constants. Notice how if the population is small then $b' \approx b$, so then we are essentially in the ideal conditions of unlimited resources. Then as N grows larger we move away from that. So, b is the same as it was in the exponential model - it is the instantaneous per capita birth rate when resources are unlimited. The constant a measures the strength of the density dependence. If a is large, then the birth rate drops sharply as the population grows. If there is no dependence on population - i.e. $a = 0$ - then we obtain the exponential model as before. So we are generalising our model. The same idea can be applied to death rates - as the population grows, death rates should increase. So,

$$d' = d + cN \tag{3}$$

for constants d and c, where d is as in the exponential model and c is the strength of the density dependence. Combining the above equations then, we have

$$\frac{dN}{dt} = ((b - aN) - (d + cN))\, N = ((b - d) - (a + c)N)\, N. \tag{4}$$

If we multiply this equation by $1 = (b - d)/(b - d)$, then we have

$$\frac{dN}{dt} = (b - d)\left(1 - \frac{(a + c)}{(b - d)}N\right) N. \tag{5}$$

As with the exponential model, let $r = b - d$ and define

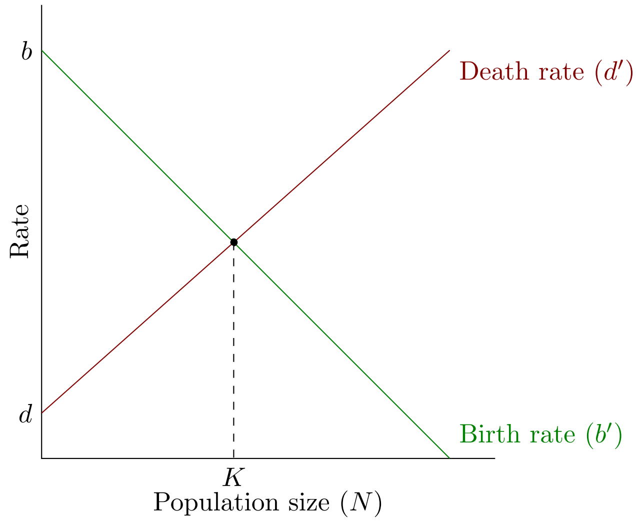
$$K = \frac{(b - d)}{(a + c)}, \tag{6}$$

called the **carrying capacity**. Then our model becomes

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right). \tag{7}$$

This is the **logistic growth equation**. It is the simplest model that describes population growth in an environment with limited resources. Notice that it looks a lot like the exponential model we have already seen, but with an extra term tacked onto it. This term, $(1 - N/K)$ represents the **unused portion of the carrying capacity**. It is the percentage of resources that are available. For example, suppose $K = 100$ and $N = 7$. Then the unused portion of the carrying capacity is $(1 - (7/100)) = 0.93$. So the population is resource rich and growing at 93% of the growth rate of an exponentially increasing population. Similarly, if N was close to K the population would grow at a much slower rate.

What happens if the population exceeds the carrying capacity? Well, the term in parentheses becomes negative and so the growth rate will be negative. So instead of growing, the population will decline and will grow again once $N < K$. With the exponential model, the population would stop growing if either r or N were zero. In the logistic model population will also stop changing if $N = K$. Other values of N will mean that the population will always tend towards K. Graphically we can see this below. The density dependent birth and death rates are plotted. There point of intersection is where $N = K$, which you can show with some simple algebra. Whenever the population size is less than K, i.e. to the left of the intersection point, births outnumber deaths and the population will increase. If the population is greater than K, i.e. to the right of the intersection point, deaths outnumber the births and the population will decrease.



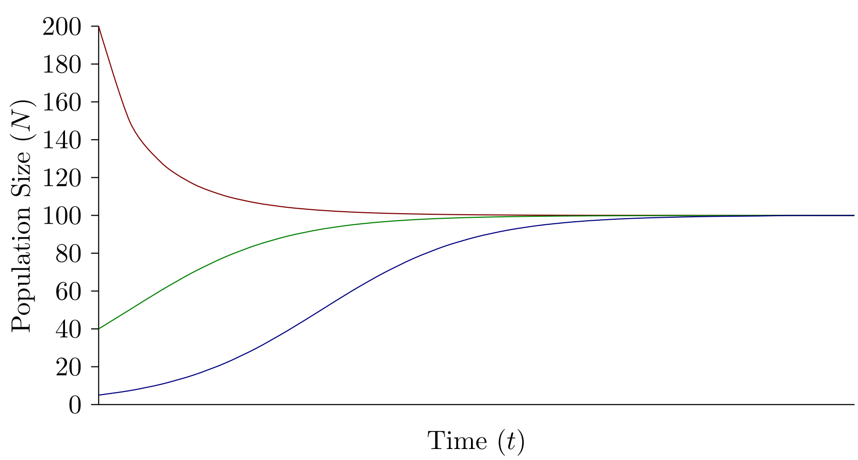
The density dependent birth and death rates intersect when the population size is equal to the carrying capacity.

The point where $N = K$ forms a **stable equilibrium**. It means as long as the starting population is non-zero, the population size will always tend towards K.

With the exponential growth model, we were able to write an explicit expression for the population size at any given time t. Although the derivation is more complicated, we can do the same with the logistic model:

$$N_t = \frac{K}{1 + (K - N_0)/N_0\, e^{-rt}}. \tag{8}$$

Graphically, the logistic growth curve looks like an S-shaped curve. The tendency towards the carrying capacity is seen as a vertical asymptote.



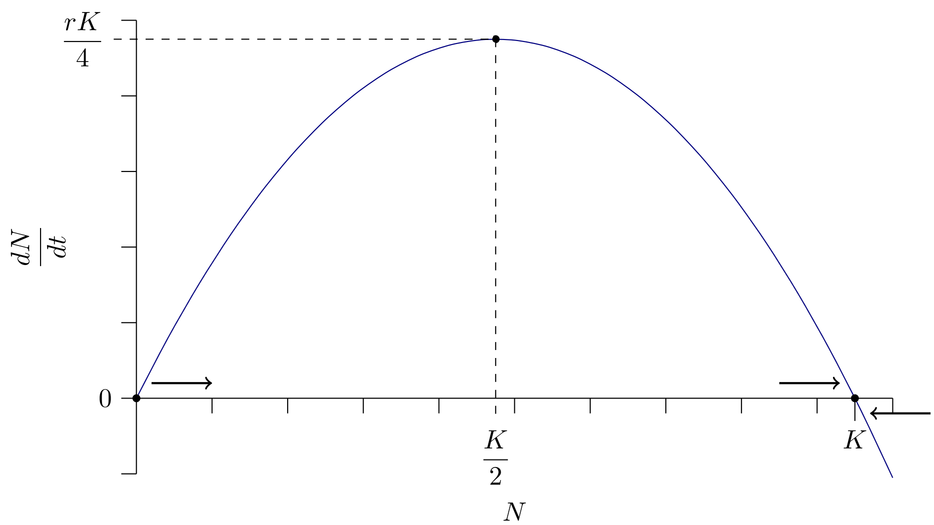
Logistic growth curves with carrying capacity $K = 100$, intrinsic growth rate $r = 0.3$ and varying initial populations.

Nobody likes change

When we study a population model we are most likely interested in **long-term behaviour**. Often, after enough time has passed, the model settles into a pattern. With respect to the previous discussion, the carrying capacity is the population that the model wants to tend towards. If it were to reach this value, then the population would remain the same. One can see this by plugging $N = K$ into our model

$$\frac{dN}{dt}\Big|_{N=K} = rN\left(1 - \frac{N}{K}\right)\Big|_{N=K} = rK\left(1 - \frac{K}{K}\right) = 0. \tag{9}$$

When the derivative is zero, there is no change. Since we assume that our change is entirely dependent on the current population size, as soon as the population remains the same for one time period, it will remain the same from that point onwards. This value $N = K$ is called an **equilibrium point**. For the logistic growth model there are two equilibrium points, $N = 0$ and $N = K$. If the population were to reach either of these values then they would never change. Since our differential equations give the derivative of our model, to find equilibrium points we simply set our equation to zero and solve.



A plot of the logistic growth model.

We can classify the equilibrium points further. Notice the arrows drawn on the plot above. When the derivative dN/dt is positive, the population N is increasing, i.e. moves to the right. Similarly when the derivative is negative the population moves to the left. These are what the arrows are indicating. When the population is *close* to zero, the population is increasing away from the equilibrium point $N = 0$. So, we call this point an **unstable** equilibrium point. When N is *close* to K, the population size is moving towards K - if its below it increases, if its above it decreases. So we call this type of equilibrium point **stable**. Alternatively we could describe these points as **repelling** of **attracting**, respectively. The population is repelled by unstable equilibrium points - it wants to move away - and is attracted to stable equilibrium points - it moves towards them.

For the case of the logistic model, there are always 2 equilibrium points - an unstable one at $N = 0$ and a stable one at $N = K$.

Symbol	Meaning
b	Instantaneous birth rate
B	Number of births
d	Instantaneous death rate
D	Number of deaths
ΔN	Change in population size between time t and $t+1$
$\frac{dN}{dt}$	Population growth rate
e	Euler's number
E	Number of emigrants leaving the population
I	Number of immigrants entering the population
K	Carrying capacity
λ	Finite rate of increase
N	Population size
N_0	Initial population
N_t	Population size at time t
r	Instantaneous rate of increase
r_d	Discrete growth factor
t	Time
t_D	Doubling time