MATH 172 - Mathematical Modelling for the Life Sciences - Spring 20
Multiple Species Population Modelling - 4 - Predator-Prey Models

A dog eat dog world

Assume we have a closed ecosystem, so that there is no migration in or out of the system. Suppose that there are only two types of animals: the predator and the prey. Here the interdependence arises because one species serves as a food source for the other species.

The general form of our **Predator-Prey Model** looks like:

$$\begin{cases} \dfrac{dP}{dt} = F_1(P) - G_1(P,Q) \\[2ex] \dfrac{dQ}{dt} = F_2(Q) + G_2(P,Q) \end{cases}$$

where the functions F_1 and F_2 represent the the population growth of P and Q, respectively, if there were no interactions between the two species. The functions G_1 and G_2 represent the affect of predation on the the two species.

For the discussion here we will look at a model where the prey population grows logistically with the absence of a predator and the prey is the sole food source for the predators. You will find examples in the homework problems of other possible predator-prey models where what we talk about here can be applied, with the obvious adjustments coming from algebraic manipulations.

We begin by obtaining the equation for the prey population, denoted by P. We assume that in the absence of predators the population follows a logistic growth model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

Now if we assume that the decrease on the number of preys is proportional to the number of predators, denoted by Q, we obtain:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - sQP,$$

where s is a positive constant. For the predator, we assume that its sole food source is the prey, so without it the population decreases exponentially. The population increase due to predation will again be assumed to be proportional to the population of prey. This yields:

$$\frac{dQ}{dt} = -uQ + vQP$$

where u and v are positive constants.

To clarify, the predator-prey model we have formulated here is

$$\begin{cases} \dfrac{dP}{dt} = rP\left(1 - \dfrac{P}{K}\right) - sQP \\[2ex] \dfrac{dQ}{dt} = -uQ + vQP \end{cases}$$

where $r, K, s, u,$ and v are all positive constants, where $u < 1$.

Equilibria and Stability

Recall that our previous encounters with determining equilibria involved us finding roots of the derivative in our model. For the predator-prey model this is still the same, the only difference is that now we require that **both derivatives need to be zero** at the same time. That is: The equilibrium points of the predator-prey model occur when

$$\frac{dP}{dt} = \frac{dQ}{dt} = 0.$$

Solving one derivative equal to zero is straightforward. Solving two simultaneously can be tricky. To facilitate this, here we introduce **nullclines**, which are lines where one of the derivatives is equal to zero.

The nullclines for $dP/dt = 0$ are found as:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - sQP = P\left[r\left(1 - \frac{P}{K}\right) - sQ\right] = 0,$$

so the two solutions are

$$P = 0 \quad \text{or} \quad Q = \frac{r}{s}\left(1 - \frac{P}{K}\right). \tag{1}$$

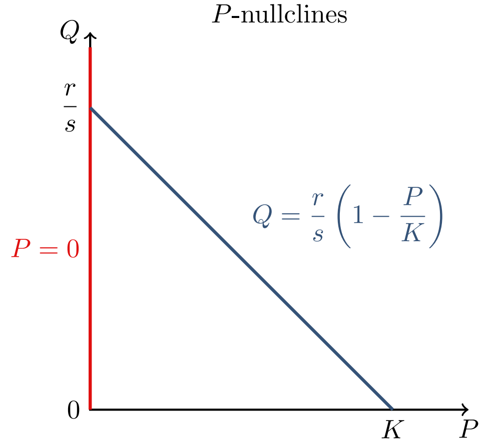
The nullclines for $dQ/dt = 0$ are found as:

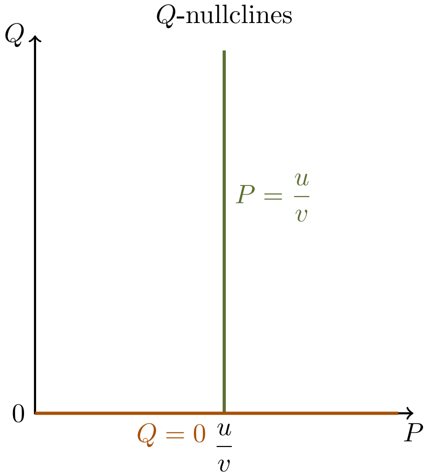
$$\frac{dQ}{dt} = -uQ + vQP = Q\left[-u + vP\right] = 0,$$

so the two solutions are

$$Q = 0 \quad \text{or} \quad P = \frac{u}{v}. \tag{2}$$

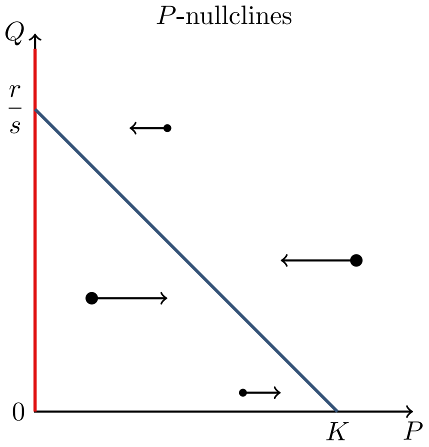
The graphs below represent plots of the solutions in (1), on the left, and (2), on the right.

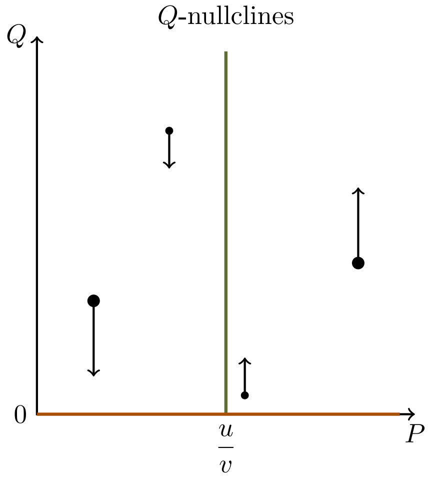




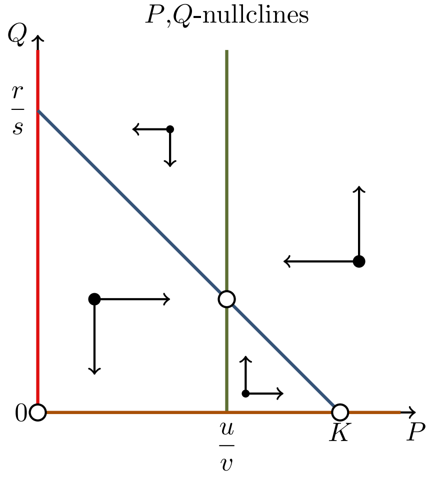
The coloured lines indicate when the given derivative is zero. We can then choose a point on the plane and from its position relative to these lines we can determine in which direction the population is moving.

For example, for the P-nullclines, if we choose a point that lies inside the triangle, then we have $Q < r\left(1 - P/K\right)/s$, so then $dP/dt > 0$ and so the population of P is increasing. Above the line $dP/dt < 0$ and so the population decreases. For the Q-nullclines, if we choose a point to the left of the line $P = u/v$, then $P < u/v$ and so $dQ/dt < 0$ and the population of Q decreases. To the right, $dQ/dt > 0$ and so the population of Q increases. We indicate these below:

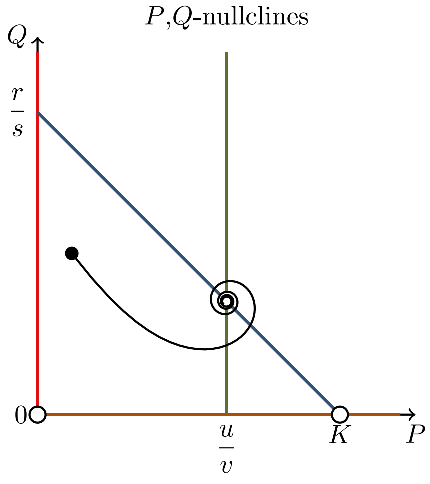




Overlaying one plot on top of the other, we obtain the following:



The equilibrium points are therefore the points of intersection of the P-nullclines with the Q-nullclines, depicted above there appears to be 3. As with single species, we can determine the stability of each of the equilibrium points by looking at the direction of the arrows with respect to the points. Taking a point anywhere on the quadrant we move in the direction the arrows tell us. The result, in the cases of a stable equilibrium, is a path that spirals into a stable equilibrium point. For example, in the diagram below, we start with a point in the 'lower left quadrant'. The arrows above tell us that we should move down and to the right. Once we cross into the 'lower right quadrant', the direction to move in changes to up and to the right. Continuing this manner we see that the result is a spiral path, similar to the one below.



Example 1: Consider the following predator-prey model:

$$\begin{cases} \dfrac{dP}{dt} = 1.3P\left(1 - \dfrac{P}{2}\right) - 0.5QP \\[2ex] \dfrac{dQ}{dt} = -0.6Q + 0.4QP. \end{cases}$$

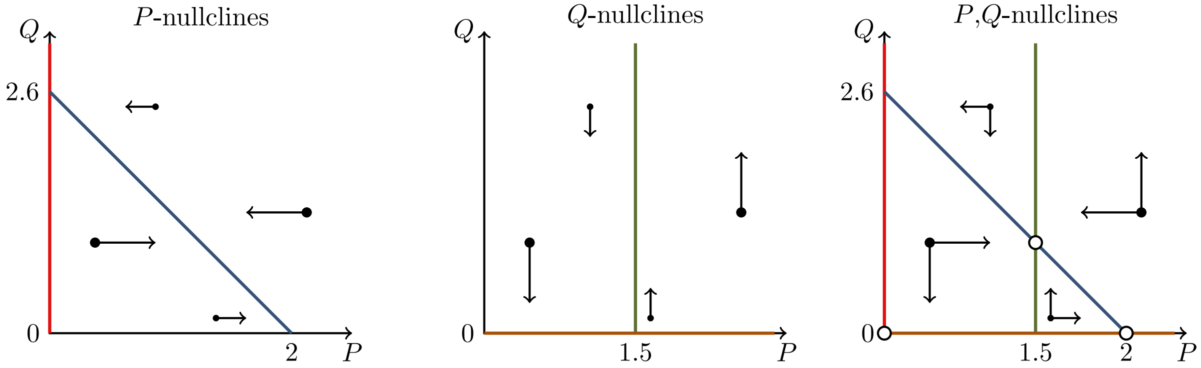
- The P-nullclines are:

$$1.3P\left(1 - \frac{P}{2}\right) - 0.5QP = P\left[1.3\left(1 - \frac{P}{2}\right) - 0.5Q\right] = 0 \implies P = 0 \text{ and } Q = 2.6 - 1.3P$$

- The Q-nullclines are:

$$-0.6Q + 0.6QP = Q\left(-0.6 + 0.4P\right) = 0 \implies Q = 0 \text{ and } P = 1.5$$

Plotting the nullclines we obtain:



To obtain the direction of the arrows on the four points indicated, we simply plug in the coordinate into each of the derivatives, dP/dt and dQ/dt, and see if they are positive or negative (increasing or decreasing). Reading in a clockwise direction starting with the bottom left, four points we could choose are $(1,1)$, $(1,2.6)$, $(2,1)$ and $(1.6,0.1)$. Remember though, we could choose *any* point in each of the sections to obtain these arrows.

- $(P,Q) = (1,1)$:

$$\frac{dP}{dt} = 1.3(1)\left(1 - \frac{1}{2}\right) - 0.5(1)(1) = 0.15 \qquad \frac{dQ}{dt} = -0.6(1) + 0.4(1)(1) = -0.2$$

P is increasing, Q is decreasing.

- $(P,Q) = (1,2.6)$:

$$\frac{dP}{dt} = 1.3(1)\left(1 - \frac{1}{2}\right) - 0.5(2.6)(1) = -0.65 \qquad \frac{dQ}{dt} = -0.6(2.6) + 0.4(2.6)(1) = -0.52$$

P is decreasing, Q is decreasing.

- $(P,Q) = (2,1)$:

$$\frac{dP}{dt} = 1.3(2)\left(1 - \frac{2}{2}\right) - 0.5(1)(2) = -1 \qquad \frac{dQ}{dt} = -0.6(1) + 0.4(1)(2) = 0.2$$

P is decreasing, Q is increasing.

- $(P,Q) = (1.6,0.1)$:

$$\frac{dP}{dt} = 1.3(1.6)\left(1 - \frac{1.6}{2}\right) - 0.5(0.1)(1.6) = 0.336 \qquad \frac{dQ}{dt} = -0.6(0.1) + 0.4(0.1)(1.6) = 0.004$$

P is increasing, Q is increasing.

We immediately see the two equilibrium points $(0, 0$ and $(2, 0)$. But as we know from our discussion on page 3, these are not stable. We want the third point that we can see is the one all the arrows are pointing to. Notice that this point is the intersection of the lines $Q = 2.6 - 1.3P$ and $P = 1.5$. So, $Q = 2.6 - 1.3(1.5) = 0.65$. Thus the stable equilibrium point is $(P, Q) = (1.5, 0.65)$.