u-Substitution

Recall the substitution rule from MATH 141 (see page 241 in the textbook).

Theorem If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

This method of integration is helpful in reversing the chain rule (Can you see why?) Let's look at some examples.

Example 1 Find $\int \sec^2(5x+1)\cdot 5\,dx$.

$$u = 5x+1$$
$$du = 5\,dx$$

$$\begin{aligned}\int \sec^2(5x+1)\cdot 5\,dx &= \int \sec^2(u)\,du \\ &= \tan(u) + C \\ &= \boxed{\tan(5x+1)+C}\end{aligned}$$

Remember, for indefinite integrals your answer should be in terms of the same variable as you start with, so remember to **substitute back in for** u.

Example 2 Evaluate the integral $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

$$u = x^2-3x+1$$
$$du = 2x-3\,dx$$

$$u = (3)^2 - 3(3) + 1 = 1$$
$$u = (5)^2 - 3(5) + 1 = 11$$

$$\begin{aligned}\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx &= \int_1^{11} \frac{1}{\sqrt{u}}\,du \\ &= \int_1^{11} u^{-1/2}\,du \\ &= 2u^{1/2}\Big|_1^{11} \\ &= 2\sqrt{11} - 2\sqrt{1} \\ &= \boxed{2\left(\sqrt{11}-1\right)}\end{aligned}$$

In the above we changed the limits of integration to coincide with our function u. Doing this means that we don't have to substitute in for u at the end like in the indefinite integral in Example 1. But if you did substitute back and use the original limits don't worry, **you get the same answer**. Try it for yourself now to see.

Example 3 Find

$$\int \frac{1}{\sqrt{8x - x^2}}\,dx.$$

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) \\ &= -((x-4)^2 - 4^2) \\ &= 4^2 - (x-4)^2 \end{aligned}$$

$$\begin{aligned} u &= x - 4 \\ du &= dx \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{8x - x^2}}\,dx &= \int \frac{1}{\sqrt{4^2 - (x-4)^2}}\,dx \\ &= \int \frac{1}{\sqrt{4^2 - (u)^2}}\,du \\ &= \sin^{-1}\left(\frac{u}{4}\right) + C \\ &= \boxed{\sin^{-1}\left(\frac{x-4}{4}\right) + C} \end{aligned}$$

Example 3 illustrates that there may not be an immediately obvious substitution. In the cases that fractions and polynomials, look at the power on the numerator. In Example 3 we had 1, so the degree was zero. To make a successful substitution, we would need u to be a degree 1 polynomial $(0 + 1 = 1)$. Obviously the polynomial on the denominator was degree 2. So we forced a degree 1 polynomial to appear by **completing the square** first.

Practice Problems

Try some of the problems below. If you get stuck, don't worry! There are hints on the next page! But do try without looking at them first, chances are you won't get hints on your exam.

1. $\int_{-1}^{1} 3x^2\sqrt{x^3+5}\,dx$
2. $\int x^3(2+x^4)^5\,dx$
3. $\int_0^7 \sqrt{4+3x}\,dx$
4. $\int \frac{1}{(1-6t)^4}\,dt$
5. $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$
6. $\int \frac{\sec(1/x)}{x^2}\,dx$
7. $\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)\,dt$
8. $\int x^2(x^3+5)^9\,dx$
9. $\int_0^1 xe^{-x^2}\,dx$
10. $\int (3t+2)^{2.4}\,dt$
11. $\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx$
12. $\int \frac{x}{(x^2+1)^2}\,dx$
13. $\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$
14. $\int e^x\sin(e^x)\,dx$
15. $\int_{-1}^{0} \frac{8x}{(4x^2+1)^2}\,dx$
16. $\int \frac{x}{x^2+1}\,dx$
17. $\int_0^1 -12x^2(4x^3-1)^3\,dx$
18. $\int \sec(2\theta)\tan(2\theta)\,d\theta$
19. $\int_{-1}^{2} 6x(x^2-1)^2\,dx$
20. $\int \sqrt{x}\sin(1+x^{3/2})\,dx$
21. $\int_0^1 \frac{24x}{(4x^2+4)^2}\,dx$
22. $\int (1+\tan(\theta))^5\sec^2(\theta)\,d\theta$
23. $\int_{-3}^{0} -\frac{8x}{(2x^2+3)^2}\,dx$
24. $\int e^{\cos(t)}\sin(t)\,dt$
25. $\int_0^1 \frac{16x}{(4x^2+4)^2}\,dx$
26. $\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$
27. $\int_{-1}^{0} 18x^2(3x^3+3)^2\,dx$
28. $\int \frac{\sin(\ln(x))}{x}\,dx$
29. $\int_0^1 -\frac{8x}{(4x^2+2)^2}\,dx$
30. $\int \frac{e^x}{e^x+1}\,dx$
31. $\int \frac{\cos(\pi/x)}{x^2}\,dx$
32. $\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$
33. $\int \frac{1}{\cos^2(t)\sqrt{1+\tan(t)}}\,dt$

Challenge Problems

Below are some harder problems that require a little more thinking/algebraic manipulation to make the substitutions work.

1. $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$
2. $\int \frac{1}{2x^2-12x+26}\,dx$
3. $\int \frac{x}{1+x^4}\,dx$
4. $\int (x+3)\sqrt{x-1}\,dx$
5. $\int \frac{x^2}{\sqrt{1-x}}\,dx$
6. $\int x^3\sqrt{x^2+1}\,dx$
7. $\int \frac{1}{\sqrt{21-4x-x^2}}\,dx$
8. $\int_{-\pi/2}^{\pi/2} \frac{x^2\sin(x)}{1+x^6}\,dx$
9. $\int \frac{3x-1}{x^2+10x+28}\,dx$
10. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$
11. $\int_{-1}^{1} \frac{\sin(x)}{1+x^2}\,dx.$
12. $\int \frac{1}{e^x+1}\,dx$

Hints to Practice Problems

1. $u = x^3 + 5$
2. $u = 2 + x^4$
3. $u = 4 + 3x$
4. $u = 1 - 6t$
5. $u = x^2$
6. $u = 1/x$
7. $u = \pi t$
8. $u = x^3 + 5$
9. $u = -x^2$
10. $u = 3t + 2$
11. $u = \sin(x)$
12. $u = x^2 + 1$
13. $u = \sin^{-1}(x)$
14. $u = e^x$
15. $u = 4x^2 + 1$
16. $u = x^2 + 1$
17. $u = 4x^3 - 1$
18. $u = 2\theta$
19. $u = x^2 - 1$
20. $u = 1 + x^{3/2}$
21. $u = 4x^2 + 4$
22. $u = 1 + \tan(\theta)$
23. $u = 2x^2 + 3$
24. $u = \cos(t)$
25. $u = 4x^2 + 4$
26. $u = \tan^{-1}(x)$
27. $u = 3x^3 + 3$
28. $u = \ln(x)$
29. $u = 4x^2 + 2$
30. $u = e^x + 1$
31. $u = \pi/x$
32. $u = \cos(x)$
33. $u = 1 + \tan(t)$

Hints to Challenge Problems

1. $x = u - 1$
2. Complete the square.
3. $u = x^2$
4. $u + 4 = x + 3$
5. $x = 1 - u$
6. $x^2 = u - 1$
7. Complete the square.
8. This is an odd function.
9. Complete the square, $x = u - 5$
10. $x = \frac{1}{2}(u - 1)$
11. This is an odd function.
12. $1 = e^x + 1 - e^x$, $u = e^x + 1$

Answers to Practice Problems

1. $4\sqrt{6} - \dfrac{16}{3}$
2. $\dfrac{1}{24}\left(2 + x^4\right)^6 + C$
3. 26
4. $\dfrac{1}{18(1-6t)^3} + C$
5. 0
6. $-\ln\left|\dfrac{1+\sin(1/x)}{\cos(1/x)}\right| + C$
7. $\dfrac{1}{\pi}$
8. $\dfrac{1}{30}\left(x^3 + 5\right)^{10} + C$
9. $\dfrac{e-1}{2e}$
10. $\dfrac{5}{51}\sqrt[5]{(3t+2)^2}\left(27t^3 + 54t^2 + 36t + 8\right) + C$
11. $1 - \cos(1)$
12. $-\dfrac{1}{2x^2+2} + C$
13. $\dfrac{1}{2}\left(\sin^{-1}(x)\right)^2 + C$
14. $-\cos(e^x) + C$
15. $-\dfrac{4}{5}$
16. $\dfrac{1}{2}\ln(x^2+1) + C$
17. -20
18. $\dfrac{\sec(2\theta)}{2} + C$
19. 27
20. $-\dfrac{3}{2}\cos\left(1_x^{3/2}\right) + C$
21. $\dfrac{3}{8}$
22. $\dfrac{1}{6}\left(\tan(\theta)+1\right)^6 + C$
23. $\dfrac{4}{7}$
24. $-e^{\cos(t)} + C$
25. $\dfrac{1}{4}$
26. $\dfrac{1}{2}\left(\tan^{-1}(x)\right)^2 + C$
27. 18
28. $-\cos(\ln(x)) + C$
29. $-\frac{1}{3}$
30. $\ln(e^x+1) + C$
31. $-\dfrac{1}{\pi}\sin\left(\dfrac{\pi}{x}\right) + C$
32. $\tan^{-1}\left(\tan^2(x/2) + C\right.$
33. $2\sqrt{1+\tan(t)} + C$

Answers to Challenge Problems

1. $\dfrac{4-2\sqrt{2}}{3}$
2. $\dfrac{1}{4}\tan^{-1}\left(\dfrac{x-3}{2}\right) + C$
3. $\dfrac{1}{2}\tan^{-1}(x^2) + C$
4. $\dfrac{2}{15}(x-1)^{3/2}(3x+17) + C$
5. $-\dfrac{2}{15}\sqrt{1-x}\left(3x^2+4x+8\right) + C$
6. $\dfrac{1}{15}(x^2+1)^{3/2}(3x^2-1) + C$
7. $\sin^{-1}\left(\dfrac{x+2}{5}\right) + C$
8. 0
9. $\dfrac{3}{2}\ln(x^2+10x+28) - \dfrac{16\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(x+5)}{3}\right) + C$
10. $\dfrac{10}{3}$
11. 0
12. $x - \ln(e^x+1) + C$