The Integral Test

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a positive, continuous, decreasing function of x. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or diverge.

Example 1: Show that the p**-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

If $p > 1$ then $f(x) = \dfrac{1}{x^p}$ is a positive, continuous, decreasing function of x. Since $\int_1^{\infty} f(x)\,dx = \dfrac{1}{p-1}$, the series converges by the Integral Test. Note that the sum of this series is not generally $\dfrac{1}{p-1}$. If $p \leq 0$, the sum diverges by the n^{th} term test. If $0 < p < 1$ then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^p}\,dx = \frac{1}{p-1}\left(\lim_{b\to\infty} b^{1-p} - 1\right) = \infty.$$

Example 2: Determine the convergence of divergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

$f(x) = xe^{-x^2}$ is positive, continuous, decreasing and $f(n) = a_n$ for all n. Further,

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^{b} xe^{-x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}\left[-e^{-b^2} - (-e^{-1})\right] = \frac{1}{2e}.$$

Since the integral converges, the series also converges.

Example 3 Determine the convergence or divergence of the series:

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}.$$

$f(x) = \dfrac{1}{x\ln(x)}$ is a positive, continuous and decreasing, and $f(n) = \dfrac{1}{n\ln(n)}$ for all n. Further,

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty} \ln\left(\ln(x)\right)\Big|_2^b = \lim_{b\to\infty} \ln\left(\ln(b)\right) - \ln(\ln(2)) = \infty.$$

Since the integral diverges, the series also diverges.

Important Note: The integral test **does not** say that the series converges to the same value as the integral. All we can determine is whether or not the series converges to **some number**. The problem of computing the value of a series is in general much harder.

Practice Problems

Use the integral test to determine whether the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$
5. $\displaystyle\sum_{n=1}^{\infty} n^{-1.4} + 3n^{-1.2}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-4n+5}$
11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$
13. $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{e^n}$
14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$
15. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

Answers to Practice Problems

1. Diverge
2. Converge
3. Diverge
4. Diverge
5. Converge
6. Converge
7. Diverge
8. Diverge
9. Converge
10. Converge
11. Converge
12. Converge
13. Converge
14. Converge
15. Converge