Interval of Convergence of Power Series

Power Series: A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **centre** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$)
3. The series converges only at $x = a$ and diverges elsewhere ($R = 0$)

The Interval of Convergence of a Power Series: The interval of convergence for a power series is the largest interval I such that for any value of x in I, the power series converges.

The interval of convergence can be calculated once you know the radius of convergence. First you solve the inequality $|x-a| < R$ for x and then you check each endpoint individually. So how do we calculate the radius of convergence? We use the ratio test (or root test) and solve.

Example 1 - Geometric Power Series: Taking all the coefficients to be 1 in the power series centred at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x.

$$\begin{aligned}
S_n &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \\
\implies (1-x)S_n &= (1-x)\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) \\
&= \left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) - \left(x + x^2 + x^3 + x^4 + x^5 \cdots + x^{n+1}\right) \\
&= 1 - x^{n+1} \\
\implies S_n &= \frac{1-x^n}{1-x}
\end{aligned}$$

So,

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} \text{ which converges if and only if } |x| < 1$$

Example 2: Consider the power series

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$$

$$\text{Centre: } a=2, \qquad c_0=1,\ c_1=-\frac{1}{2},\ c_2=\frac{1}{4},\ \ldots,\ c_n=\left(-\frac{1}{2}\right)^n,$$

$$\text{Ratio: } r=\frac{c_{n+1}(x-2)^{n+1}}{c_n(x-2)^n}=\frac{c_1(x-2)}{c_0}=\frac{-\frac{1}{2}(x-2)}{1}=-\frac{x-2}{2}$$

The series converges when $|r|<1$, that is,

$$\left|-\frac{x-2}{2}\right|<1\Longrightarrow\left|\frac{x-2}{2}\right|<1\Longrightarrow|x-2|<2\Longrightarrow-2<x-2<2\Longrightarrow 0<x<4.$$

Example 3: For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|(-1)^n\frac{x^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n-1}x^n}\right|=\left|\frac{nx}{x+1}\right|=|x|\frac{n}{n+1}\overset{n\to\infty}{\longrightarrow}|x|$$

The series converges absolutely when $|x|<1$ and diverges when $|x|>1$. It remains to see what happens at the endpoints; $x=-1$ and $x=1$.

$x=-1$: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}=\sum_{n=1}^{\infty}\frac{-1}{n}=-\sum_{n=1}^{\infty}\frac{1}{n}\Longrightarrow$ the series diverges at $x=-1$.

$x=1$: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=$ the Alternating Harmonic Series $\Longrightarrow$ the series converges at $x=1$.

So, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $-1<x\leq 1$ and diverges elsewhere.

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\left|\frac{x^{n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{x^n}\right|=\left|\frac{x}{n+1}\right|=\frac{|x|}{n+1}\overset{n\to\infty}{=\!\!\longrightarrow}0$$

Since the value of the limit is 0, no matter what real number we choose for x and $0<1$, the series converges absolutely for all values of x. $\quad(x\in\mathbb{R},\ \ -\infty<x<\infty,\ \ (-\infty,\infty))$.

Fact: There is always at least one point for which a power series converges: the point $x=a$ at which the series is centred.

Example 4: Find the interval and radius of convergence for

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}.$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^{3/2}3^{n+1}} \cdot \frac{n^{3/2}3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{xn^{3/2}}{(n+1)^{3/2}3} \right| = \frac{|x|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{3/2} = \frac{|x|}{3}.$$

So the series converges absolutely when $\frac{|x|}{3} < 1 \implies |x| < 3 \implies -3 < x < 3$.

Check the endpoints:

$x = -3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ which is an alternating p-series with $p = \frac{3}{2}$, so it *converges.*

$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \frac{3}{2}$, so it *converges.*

Thus the interval of convergence is $[-3, 3]$ and the radius of convergence is $R = 3$.

Practice Problems

Determine the interval of convergence of the following power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} x^n$
3. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
7. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$
11. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$

Answers to Practice Problems

1. $-1 < x \leq 1$
2. $-1 < x < 1$
3. $x = 0$
4. $-\frac{1}{10} \leq x \leq \frac{1}{10}$
5. $-5 \leq x \leq 5$
6. $-\infty < x < \infty$
7. $2 < x \leq 4$
8. $-5 < x < 3$
9. $-\frac{1}{3} \leq x < \frac{5}{3}$
10. $3 \leq x \leq 5$
11. $x = \frac{1}{2}$
12. $-\infty < x < \infty$