# Improper Integrals

**Definition 1**: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.
2. If $f(x)$ is continuous on $(-\infty, b]$, then $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty}\int_{-a}^b f(x)\,dx$.
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$, where $c$ is any real number.

**Definition 2**: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at $a$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^a f(x)\,dx$.
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at $b$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$.
3. If $f(x)$ is discontinuous at $c \in (a, b)$ and continuous elsewhere, then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

In each case, if the limit is finite we sat that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

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**Example 1**:Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

$$u = \ln(x) \qquad dv = \frac{1}{x^2}\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$$

$$\begin{aligned}
\int_1^b \frac{\ln(x)}{x^2}\,dx &= -\frac{\ln(x)}{x}\bigg|_1^b - \int_1^b -\frac{1}{x^2}\,dx \\
&= -\frac{\ln(x)}{x} - \frac{1}{x}\bigg|_1^b \\
&= -\frac{\ln(b)}{b} - \frac{1}{b} - \left[-\frac{\ln(1)}{1} - \frac{1}{1}\right] \\
&= -\frac{\ln(b)}{b} - \frac{1}{b} + 1
\end{aligned}$$

Now we take a limit,

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b} - \frac{1}{b} + 1\right] = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b}\right] - 0 + 1 \overset{\text{L'H}}{=} \lim_{b\to\infty}\left[-\frac{1/b}{1}\right] + 1 = 0 + 1 = \boxed{1}$$

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**L'Hôpital's Rule** Suppose that $f(a) = g(a) = 0$, that $f(x)$ and $g(x)$ are differentiable on an open interval $I$ containing $a$ and that $g'(x) \neq 0$ on $I$ if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the left and right both exist.

**Example 2**: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

According to part 3 of Definition 1, we can choose any real number $c$ and split this integral into two integrals and then apply parts 1 and 2 to each piece. Let's choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Now we will evaluate each piece separately.

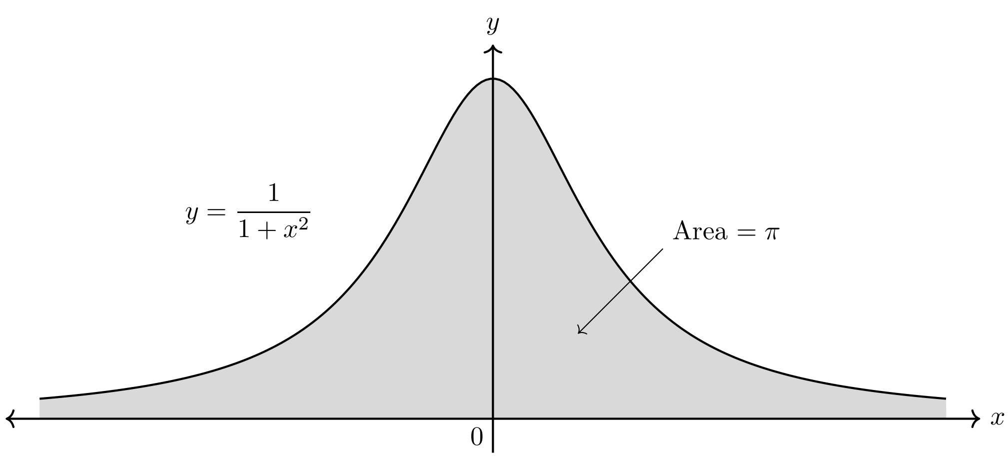
$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx \\ &= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_{1}^{0} \\ &= \lim_{a\to-\infty} \tan^{-1}(0) - \tan^{-1}(a) \\ &= \lim_{a\to-\infty} -\tan^{-1}(a) \\ &= \frac{\pi}{2},\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \tan^{-1}(x)\Big|_{0}^{b} \\ &= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(0) \\ &= \lim_{b\to\infty} \tan^{-1}(b) \\ &= \frac{\pi}{2}.\end{aligned}$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Since $1/(1+x^2) > 0$ on $\mathbb{R}$, the improper integral can be interpreted as the (finite) area between the curve and the $x$-axis.




Note that since we split our limits of integration at 0 and we took limits to both $\pm\infty$, we can do everything in one step, as follows:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{c\to\infty} \int_{-c}^{c} \frac{1}{1+x^2}\,dx = \lim_{c\to\infty} \tan^{-1}(x)\Big|_{-c}^{c} = \lim_{c\to\infty} \tan^{-1}(c) - \tan^{-1}(-c) = \lim_{c\to\infty} 2\tan^{-1}(c) = 2\frac{\pi}{2} = \boxed{\pi}$$

**Example 3**: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

First we find the integral over the region $[a, 1]$ where $0 < a \leq 1$.

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_a^1 = 2\sqrt{x}\Big|_a^1 = 2 - 2\sqrt{a} = 2(1-\sqrt{a}).$$

Then we find the limit as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\left(1-\sqrt{a}\right) = 2.$$

Therefore,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$$

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**A Special Example**: For what values of $p$ does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

We split this investigation into two cases; when $p \neq 1$ and when $p = 1$.

If $p \neq 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^b \\ &= \lim_{b\to\infty} \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\Bigg|_1^b \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases}\frac{1}{p-1}, & p>1 \\ \infty, & p<1.\end{cases}\end{aligned}$$

If $p = 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln(x)\Big|_1^b \\ &= \lim_{b\to\infty} [\ln(b) - \ln(1)] \\ &= \lim_{b\to\infty} \ln(b) = \infty\end{aligned}$$

Combining these two results we have

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\frac{1}{p-1}, & p>1 \\ \infty, & p\leq 1\end{cases}}$$

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## Practice Problems

Try some of the problems below.

1. $\int_1^\infty \frac{1}{(3x+1)^2}\,dx$
2. $\int_{-\infty}^0 \frac{1}{2x-5}\,dx$
3. $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$
4. $\int_{-\infty}^{-1} e^{-2t}\,dt$
5. $\int_{-\infty}^\infty 2-v^4\,dv$
6. $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$
7. $\int_{-\infty}^\infty \cos(\pi t)\,dt$
8. $\int_0^\infty \frac{1}{z^2+3z+2}\,dz$
9. $\int_{-\infty}^\infty x^3 e^{-x^4}\,dx$
10. $\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$
11. $\int_0^\infty \frac{x\tan^{-1}(x)}{(1+x^2)^2}\,dx$
12. $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$
13. $\int_6^8 \frac{4}{(x-6)^3}\,dx$
14. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$
15. $\int_0^1 \frac{1}{4y-1}\,dy$
16. $\int_{\pi/2}^\pi \csc(x)\,dx$
17. $\int_0^1 \frac{e^{1/x}}{x^3}\,dx$
18. $\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx$

## Answers to Practice Problems

1. $\frac{1}{12}$
2. Diverges
3. $\frac{1}{4}$
4. Diverges
5. Diverges
6. $\frac{2}{e}$
7. Diverges
8. $\ln(2)$
9. 0
10. $\frac{\sqrt{3}\pi}{9}$
11. $\frac{\pi}{8}$
12. 2
13. Diverges
14. $\frac{\pi}{2}$
15. Diverges
16. Diverges
17. Diverges
18. 0