Computing Taylor Series

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by** $f(x)$ **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin Series generated by** $f(x)$ is the Taylor series generated by $f(x)$ at $a = 0$.

Example 1: Find the Taylor series generated by $f(x) = \dfrac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\dfrac{1}{x}$?

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\frac{1}{x}$	$\frac{1}{2}$
1	$(-1) \cdot \frac{1}{x^2}$	$(-1)\frac{1}{2^2}$
2	$(-1)^2 \cdot \frac{2 \cdot 1}{x^3}$	$(-1)^2 \frac{2 \cdot 1}{2^3}$
3	$(-1)^3 \cdot \frac{3 \cdot 2 \cdot 1}{x^4}$	$(-1)^3 \frac{3 \cdot 2 \cdot 1}{2^4}$
4	$(-1)^4 \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{x^5}$	$(-1)^4 \frac{4 \cdot 3 \cdot 2 \cdot 1}{2^5}$
n	$(-1)^n \cdot \frac{n!}{x^{n+1}}$	$(-1)^n \frac{n!}{2^{n+1}}$

The key thing to do when looking for the general term is to not simplify everything. You should try and only group those terms that come from the "same place." For example, when $n = 2$ we could have cancelled a 2 from the numerator and denominator of $f''(2)$. But since the 2 in the numerator came from differentiating and the 2 on the denominator came from plugging in $x = a$, we leave them alone. Leaving factors alone this way will help you more easily see where each number in the factor is coming from and its relation to the value of n.

So, the Taylor Series generated by $f(x) = \dfrac{1}{x}$ centred at $a = 2$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{n!}{2^{n+1}}}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n}$$

Note that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + \frac{(-1)^n(x-2)^n}{2^{n+1}}$$

is geometric with first term $\dfrac{1}{2}$ and ratio $r = -\dfrac{(x-2)}{2}$. So it converges (absolutely) for

$$\left| -\frac{(x-2)}{2} \right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow 0 < x < 4.$$

Now we check the endpoints:

$x = 0$: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(0-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2} \Longrightarrow \text{diverges by } n\text{th term test.}$$ (Also clear since $f(x) = \frac{1}{x}$ is not defined at $x = 0$)

$x = 4$: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(4-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \Longrightarrow \text{diverges by } n\text{th term test.}$$

Thus the only values of x for which this Taylor Series converges are $\boxed{0 < x < 4}$.

Example 2: Find the Taylor Series generated by $f(x) = \cos(x)$ at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
$2n$	$(-1)^n \cos(x)$	$(-1)^n$
$2n+1$	$(-1)^{n+1} \sin(x)$	$(-1)^n 0$

When terms are alternating between 0s and non-zero terms, take a look at the parity of the values of n for which they appear. That is, check if all the 0s occur when n is odd (or when n is even). Once you figure out which is which you can ignore all the zero terms by considering $2n$ or $2n+1$ instead of just n.

If you are dealing with trigonometric functions, it is likely that at some point there will be some repetition happening. For example here $f^{(4)}(x) = f(x)$. So then you might be able to see what is happening by only using the terms up until the repeat.

So the Taylor Series generated by $f(x) = \cos(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$

To find the interval of convergence, we can use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right| = \frac{x^2}{(2n+2)(2n+1)} \overset{n\to\infty}{\longrightarrow} 0$$

So this Taylor Series converges for all $x \in \mathbb{R}$.

Example 3: Find the Taylor Series generated by $f(x) = e^x$.

Note that $f^{(n)}(x) = f(x) = e^x$ for every positive integer n. So $f^{(n)}(0) = e^0 = 1$ for each n, so then the Taylor Series generated by $f(x) = e^x$ at $a = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

One then verifies that it converges for all $x \in \mathbb{R}$.

Practice Problems

Find the Taylor Series generated by the following functions at the given centre. Also find the interval of absolute convergence of the Taylor Series.

1. $f(x) = \dfrac{1}{(1-x)^2}, \quad a = 0$
2. $f(x) = x^4 - 3x^2 + 1, \quad a = 1$
3. $\ln(1+x), \quad a = 0$
4. $f(x) = x - x^3, \quad a = -2$
5. $\sin(\pi x), \quad a = 0$
6. $f(x) = e^x, \quad a = 3$
7. $\cos(3x), \quad a = 0$
8. $f(x) = \dfrac{1}{x}, \quad a = -3$
9. $f(x) = e^{5x}, \quad a = 0$
10. $f(x) = \cos(x), \quad a = \pi$
11. $f(x) = xe^x, \quad a = 0$
12. $f(x) = \sin(x), \quad a = \dfrac{\pi}{2}$
13. $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 9$
14. $f(x) = \dfrac{1}{x^2}, \quad a = 1$
15. $f(x) = \cos(x^2), \quad a = 0$

Answers to Practice Problems

1. $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n, \quad |x| < 1$
2. $-1-2(x-1)+3(x-1)^2+4(x-1)^3+(x-1)^4, \quad |x| < \infty$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n, \quad |x| < 1$
4. $6 - 11(x+2) + 6(x+2)^2 - (x+2)^3, \quad |x| < \infty$
5. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}x^{2n+1}, \quad |x| < \infty$
6. $\displaystyle\sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n, \quad |x| < \infty$
7. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}}{(2n)!}x^{2n}, \quad |x| < \infty$
8. $\displaystyle -\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}(x+3)^n, \quad |x+3| < 3$
9. $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n, \quad |x| < \infty$
10. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}, \quad |x| < \infty$
11. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n-1)!}x^n, \quad |x| < \infty$
12. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}, \quad |x| < \infty$
13. $\displaystyle\frac{1}{3}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n, \quad |x-9| < 9$
14. $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n, \quad |x-1| < 1$
15. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}, \quad |x| < \infty$