Direct and Limit Comparison Tests

We have seen that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. We now apply the same idea to infinite series instead.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, for some N, then,

1. If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and $L > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 1: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ converges or diverges.

We have $\frac{1}{2^n + n} \leq \frac{1}{2^n}$ for all $n \geq 1$. So,

$$\sum_{n=1}^{\infty} \frac{1}{2^n + n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Since the series on the right is a geometric series with $r = \frac{1}{2}$, it converges. So we determine that our series of interest also converges.

Example 2: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$ converges or diverges.

We have $\frac{1}{n^2 + n + 1} \leq \frac{1}{n^2}$ for all $n \geq 1$. So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the series on the right is a p series with $p = 2 > 1$, it converges. So we determine that our series of interest also converges.

Example 3: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 - 1}}{n^5 + 3}$ converges or diverges.

We have $\frac{\sqrt{n^4 - 1}}{n^5 + 3} < \frac{\sqrt{n^4}}{n^5 + 3} = \frac{n^2}{n^5 + 3} < \frac{n^2}{n^5} = \frac{1}{n^3}$. So,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4 - 1}}{n^5 + 3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Since the series on the right is a p series with $p = 3 > 1$, it converges. So we determine that our series of interest also converges.

Example 4: Determine whether the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-5}$ converge or diverges.

We can see that the direct comparison test will not work here. So let's try the limit comparison test. We have $\dfrac{1}{n^2-1} \approx \dfrac{1}{n^2}$. Further,

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-5} = \lim_{n\to\infty} 1 + \frac{5}{n^2-5} = 1.$$

Since $\sum_{n=3}^{\infty} \dfrac{1}{n^2}$ is a p-series with $p = 2 > 1$, it converges. Thus by the limit comparison test, $\sum_{n=5}^{\infty} \dfrac{1}{n^2-5}$ converges also.

Example 5: Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ converges or diverges.

We have $\dfrac{n^3-2n}{n^4+3} \approx \dfrac{n^3}{n^4} = \dfrac{1}{n}$. Further,

$$\lim_{n\to\infty} \frac{\frac{n^3-2n}{n^4+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n^3-2n)}{n^4+3} = \lim_{n\to\infty} \frac{n^4-2n^2}{n^4+3} = \lim_{n\to\infty} 1 - \frac{3+2n^2}{n^4+3} = 1.$$

Since $\sum_{n=2}^{\infty} \dfrac{1}{n}$ is a p-series with $p = 1$, it diverges. Thus by the limit comparison test, $\sum_{n=2}^{\infty} \dfrac{n^3-2n}{n^4+3}$ diverges also.

Practice Problems

Using one of the comparison tests, determine whether the following series converge or diverge.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin(n)}{10^n}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$
10. $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$
11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{1/n}}{n}$
14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Hints to Practice Problems

1. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$
2. Compare with $\sum_{n=2}^{\infty} \frac{1}{n^{1.5}}$
3. Compare with $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$
4. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
5. Compare with $\sum_{n=0}^{\infty} \frac{2}{10^n}$
6. Compare with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$
7. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$
8. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$
9. Compare with $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$
10. Compare with $\sum_{n=3}^{\infty} \frac{1}{n^2}$
11. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$
12. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$
13. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$
14. Compare with $\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^n$
15. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

Answers to Practice Problems

1. Diverges
2. Converges
3. Diverges
4. Converges
5. Converges
6. Diverges
7. Converges
8. Diverges
9. Converges
10. Converges
11. Diverges
12. Converges
13. Diverges
14. Converges
15. Diverges