Direct and Limit Comparison Tests

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Why does this make sense?

1. If the area under the curve of $g(x)$ is *finite* and $f(x)$ is bounded above by $g(x)$ (and below by 0), then the area under the curve of $f(x)$ must be *less than or equal to* the area under the curve of $g(x)$. A positive number less that a *finite* number is also *finite*.
2. If the area under the curve of $f(x)$ is *infinite* and $g(x)$ is bounded below by $f(x)$, then the area under the curve of $g(x)$ must be *"less than or equal to"* the area under the curve of $g(x)$. Since there is no finite number "greater than" infinity, the area under $g(x)$ must also be *infinite*.

Example 1: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

We want to find a function $g(x)$ such that for some $a \in \mathbb{R}$, $f(x) = \dfrac{\cos^2(x)}{x^2} \leq g(x)$ or $f(x) = \dfrac{\cos^2(x)}{x^2} \geq g(x)$ for all $x \geq a$. One way we can do this is by finding *bounds* for $f(x)$. Since $0 \leq \cos^2(x) \leq 1$ for all x,

$$\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

So then we can use $g(x) := \dfrac{1}{x^2}$. So,

$$0 \leq \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx \leq \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

So $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.

Example 2: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Since $x \geq x - e^{-x}$, $f(x) := \dfrac{1}{x} \leq \dfrac{1}{x - e^{-1}} =: g(x)$ for all $x \geq 3$. So,

$$0 \leq \int_3^\infty f(x)\,dx \leq \int_3^\infty g(x)\,dx.$$

By the Direct Comparison Test then, $\displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,dx$ *diverges* since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges.

Important Note: The direct comparison test **does not** say that the two integrals converge to the same number. The test only tells you whether or not both integrals converge or diverge.

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Why does this make sense? The convergence is really only dependent on the "tail" of the integral. That is, the convergence is dictated by what happens "at infinity." If for sufficiently large values of x, $f(x) \approx Lg(x)$ and one of the two integrals converges, then the other one should also converge, since it is only off by "about a scalar multiple." The same goes for diverging, if one diverges, then multiplying it by a positive number won't suddenly make it converge, so the other one should also diverge.

Example 3: Show that

$$\int_1^\infty \frac{1}{1 + x^2}\,dx$$

converges.

Let $f(x) := \dfrac{1}{1 + x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1 + x^2} = \lim_{x\to\infty} \frac{1 + x^2 - 1}{1 + x^2} = \lim_{x\to\infty} \left(1 - \frac{1}{1 + x^2}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1}{1 + x^2}\,dx$ *converges.*

Example 4: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.

Let $f(x) := \dfrac{1-e^{-x}}{x}$ and $g(x) := \dfrac{1}{x}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(1-e^{-x}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ *diverges*.

Important Note: The limit comparison test **does not** tell you the value of either integral. The value of the integral you are interested in **is not** equal to L and **is not** equal to the value of the integral you compare it to. This test **only tells you about convergence/divergence**.

Practice Problems

Using one of the comparison tests, determine whether or not the following integrals converge or diverge.

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Answers to Practice Problems

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