Absolute Convergence and More Tests

Definition A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- $\lim_{n\to\infty} b_n = 0$.

The Ratio and Root Tests: Let $\sum a_n$ be any series and suppose

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$

or

Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using these tests.

A Summary:

- Absolute converge is a *stronger* type of convergence than regular convergence. So absolute convergence *implies* convergence, but not the other way around.
- An example of a conditionally convergent series is the **alternating harmonic series** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$. This series converges, by the alternating series test, but the series $\sum_{n=1}^{\infty}\frac{1}{n}$ (called the **harmonic series**) is one that we know to diverge, by the integral test (or as you should recognise it, its the p-series with $p = 1$).
- The alternating series test, like the nth term test is one you can quickly use to try and save yourself some work. As soon as you see an alternating sign ($(-1)^n$ or $(-1)^{n+1}$ for example) you can check the limit. If the limit is not obvious then maybe you want to try a different test.
- The ratio test is going to be your best friend for any series that involve *factorials*. We now recall what factorials are: $n!$ (read "n factorial") is the product of all positive integers less than or equal to n. That is,

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1,$$

where it is a convention that $0! = 1$. For example $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. Factorials will come up a lot in the rest of the material for this class so it is important to get comfortable with them. You should especially be comfortable with dividing different factorials. Any time you see factorials in a series, you should think "Ratio Test."

- The alarm to use the root test is if there are a lot of nth powers of things in your series. The root test will simplify the limit you look at by eliminating these powers.
- It is common that given a random series both the root or ratio test will be viable options for you to use. In this case you should use whichever one you are more comfortable with.

Now lets see some examples using these tests.

Example 1: Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges or diverges.

If we consider $f(x) = \dfrac{x^2}{x^3+1}$, then $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2}$. Thus $f(x)$ is decreasing when $x > \sqrt[3]{2}$. So, the sequence $\dfrac{n^2}{n^3+1}$ is decreasing when $n \geq 2$. This satisfies the first condition of the alternating series test. For the second,

$$\lim_{n\to\infty}\frac{n^2}{n^3+1} = \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0.$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges by the alternating series test.

Example 2: Determine whether the series $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}$ converges or diverges.

We saw a factorial so we will of course be using the ration test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\underbrace{\frac{(2(n+1))!}{((n+1)!)^2}}_{a_{n+1}} \cdot \underbrace{\frac{(n!)^2}{(2n)!}}_{1/a_n}\right| = \frac{(2n+2)!}{(n+1)!\cdot(n+1)!}\,\frac{n!\cdot n!}{(2n)!} = \frac{(2n+2)\cdot(2n+1)\cdot(2n)!}{(n+1)\cdot n!\cdot(n+1)\cdot n!}\,\frac{n!\cdot n!}{(2n)!} = \frac{(2n+2)\cdot(2n+1)}{(n+1)\cdot(n+1)}$$

(You should check through this line carefully to make sure you understand where and why all the cancellations occur). So,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4 > 1.$$

Thus the series $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the ratio test.

Example 3: Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^{1-3n}}{4^{2n}}$ converges or diverges.

Both the numerator and denominator contain nth powers, so lets use the root test. We have,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{n^{1-3n}}{4^{2n}}\right|} = \sqrt[n]{\frac{n\cdot n^{-3n}}{2^{2n}}} = \frac{\sqrt[n]{n}n^{-3}}{4^n}.$$

It is at this point we will recall the common limit: $\displaystyle\lim_{n\to\infty}\sqrt[n]{n} = 1$. So,

$$\lim_{n\to\infty}\frac{\sqrt[n]{n}n^{-3}}{4^2} = \frac{1\cdot 0}{4^2} = 0 < 1.$$

Thus the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^{1-3n}}{4^{2n}}$ converges absolutely by the root test.

Practice Problems

Using the alternating series, ratio or root test, determine whether or not the following series converge absolutely, converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+2}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$
10. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$
11. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$
12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

Answers to Practice Problems

1. Converges
2. Diverges
3. Converges absolutely
4. Converges absolutely
5. Converges
6. Converges absolutely
7. Converges
8. Converges absolutely
9. Converges absolutely
10. Diverges
11. Converges
12. Converges