Section 8.1: Using Basic Integration Formulas

A Review: The basic integration formulas summarise the forms of indefinite integrals for may of the functions we have studied so far, and the substitution method helps us use the table below to evaluate more complicated functions involving these basic ones. So far, we have seen how to apply the formulas directly and how to make certain u-substitutions. Sometimes we can rewrite an integral to match it to a standard form. More often however, we will need more advanced techniques for solving integrals. First, let's look at some examples of our known methods.

Basic integration formulas			
1. $\displaystyle\int k\,dx = kx + C$	(any number k)	12. $\displaystyle\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$(n \neq -1)$	13. $\displaystyle\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$	
3. $\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$		14. $\displaystyle\int \sec(x)\,dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$	
4. $\displaystyle\int e^x\,dx = e^x + C$		15. $\displaystyle\int \csc(x)\,dx = -\ln\lvert\csc(x) + \cot(x)\rvert + C$	
5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$	$(a > 0,\ a \neq 1)$	16. $\displaystyle\int \sinh(x)\,dx = \cosh(x) + C$	
6. $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$		17. $\displaystyle\int \cosh(x)\,dx = \sinh(x) + C$	
7. $\displaystyle\int \cos(x)\,dx = \sin(x) + C$		18. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
8. $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$		19. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
9. $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C$		20. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$	$(a > 0)$
10. $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$		21. $\displaystyle\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
11. $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C$		22. $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$	$(x > a > 0)$

Example 1 - Substitution: Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

$u = x^2 - 3x + 1$

$du = 2x - 3\,dx$

$u = (3)^2 - 3(3) + 1 = 1$

$u = (5)^2 - 3(5) + 1 = 11$

$$\begin{aligned}\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx &= \int_1^{11} \frac{1}{\sqrt{u}}\,du \\ &= \int_1^{11} u^{-1/2}\,du \\ &= 2u^{1/2}\Big|_1^{11} \\ &= 2\sqrt{11} - 2\sqrt{1} \\ &= \boxed{2\left(\sqrt{11}-1\right)}\end{aligned}$$

Example 2 - Complete the Square: Find

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx.$$

$$\begin{aligned}8x - x^2 &= -(x^2 - 8x) \\ &= -((x-4)^2 - 4^2) \\ &= 4^2 - (x-4)^2\end{aligned}$$

$u = x - 4$

$du = dx$

$$\begin{aligned}\int \frac{1}{\sqrt{8x-x^2}}\,dx &= \int \frac{1}{\sqrt{4^2-(x-4)^2}}\,dx \\ &= \int \frac{1}{\sqrt{4^2-(u)^2}}\,du \\ &= \sin^{-1}\left(\frac{u}{4}\right) + C \\ &= \boxed{\sin^{-1}\left(\frac{x-4}{4}\right) + C}\end{aligned}$$

Example 3 - Trig Identities: Calculate

$$\int \cos(x)\sin(2x) + \sin(x)\cos(2x)\,dx.$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$\begin{aligned}\int \cos(x)\sin(2x) + \sin(x)\cos(2x)\,dx &= \int \sin(x+2x)\;dx \\ &= \int \sin(3x)\;dx \\ &= \int \frac{1}{3}\sin(u)\;du \\ &= -\frac{1}{3}\cos(u) + C \\ &= \boxed{-\frac{1}{3}\cos(3x) + C}\end{aligned}$$

Example 4 - Trig Identities: Find

$$\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\, dx.$$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\, dx &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\sin(x)}{1-\sin^2(x)}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(x)} + \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}\, dx \\
&= \int_0^{\frac{\pi}{4}} \sec^2(x) + \sec(x)\tan(x)\, dx \\
&= \tan(x) + \sec(x)\Big|_0^{\frac{\pi}{4}} \\
&= \tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) - (\tan(0) + \sec(0)) \\
&= 1 + \sqrt{2} - (0+1) \\
&= \boxed{\sqrt{2}}
\end{aligned}$$

Example 5 - Clever Substitution Evaluate

$$\int \frac{1}{(1+\sqrt{x})^3}\, dx.$$

$$\begin{aligned}
u &= 1+\sqrt{x} \\
du &= \frac{1}{2\sqrt{x}}\, dx \\
2\sqrt{x}\, du &= dx \\
2(u-1)\, du &= dx
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{(1+\sqrt{x})^3}\, dx &= \int \frac{2(u-1)}{u^3}\, du \\
&= \int \frac{2}{u^2} - \frac{2}{u^3}\, du \\
&= \int 2u^{-2} - 2u^{-3}\, du \\
&= -2u^{-1} + u^{-2} + C \\
&= -\frac{2}{u} + \frac{1}{u^2} + C \\
&= \boxed{-\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C}
\end{aligned}$$

Example 6 - Properties of Trig Integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos(x)\, dx.$$

$$f(x) = x^3 \implies f(-x) = (-x)^3 = -x^3 = -f(x)$$
$$\implies x^3 \text{ is an odd function}$$

$$g(x) = \cos(x) \implies f(-x) = \cos(-x) = \cos(x) = f(x)$$
$$\implies x^3 \text{ is an even function}$$

Putting these two facts together we see that $x^3\cos(x)$ is an odd function and is symmetric over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus (by Theorem 8, Section 5.6)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos(x)\, dx = \boxed{0}$$

Section 8.2: Techniques of Integration

A New Technique: Integration is a technique used to simplify integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when one of the functions ($f(x)$ or $g(x)$) can be differentiated repeatedly and the other function can be integrated repeatedly without difficulty. The following are two such integrals:

$$\int x\cos(x)\,dx \text{ and } \int x^2 e^x\,dx.$$

Notice $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly (they are even eventually zero) and $g(x) = \cos(x)$ and $g(x) = e^x$ can be integrated repeatedly without difficulty.

An Application of the Product Rule: If $f(x)$ and $g(x)$ are differentiable functions of x, the product rule says that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides and rearranging gives us the **Integration by Parts** formula!

$$\begin{aligned}
&\int \frac{d}{dx}\left[f(x)g(x)\right]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \\
\Longrightarrow\quad &\int f(x)g'(x)\,dx = \int \frac{d}{dx}\left[f(x)g(x)\right]\,dx - \int f'(x)g(x)\,dx \\
\Longrightarrow\quad &\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}
\end{aligned}$$

In differential form, let $u = f(x)$ and $v = g(x)$. Then,

Integration by Parts Formula:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Remember, all of the techniques that we talk about are supposed to make integrating easier! Even though this formula expresses one integral in terms of a second integral, the idea is that the second integral, $\int v\,du$, is easier to evaluate. The key to integration by parts is making the right choice for u and v. Sometimes we may need to try multiple options before we can apply the formula.

Example 1: Find

$$\int x\cos(x)\,dx.$$

We have to decide what to assign to u and what to assign to dv. Our goal is to make the integral *easier*. One thing to bear in mind is that whichever term we let equal u we need to differentiate - so if differentiating makes a part of the integrand simpler that's probably what we want! In this cases differentiating $\cos(x)$ gives $-\sin(x)$, which is no easier to deal with. But differentiating x gives 1 which *is* simpler. So we have,

$$\begin{aligned} u &= x & dv &= \cos(x)\,dx \\ du &= dx & v &= \sin(x) \end{aligned}$$

$$\begin{aligned} \int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx \\ &= \boxed{x\sin(x) + \cos(x) + C} \end{aligned}$$

Example 2: Evaluate

$$\int x^2 e^x\,dx.$$

Here we go through the same thought process. If $u = e^x$ then $du = e^x\,dx$, which doesn't make the problem any easier (though it doesn't make it any harder either). But in this case $dv = x^2$ would give $v = \frac{1}{3}x^3$ which arguably is *not* simpler that x^2. So,

$$\begin{aligned} u &= x^2 & dv &= e^x\,dx \\ du &= 2x\,dx & v &= e^x \end{aligned}$$

$$\int x^2 e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

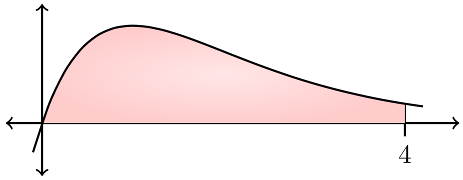
It's at this point we see that we still cannot integrate the integral on the write easily. This is okay. **Sometimes we may have to apply the integration by parts formula more than once!**

$$\begin{aligned} u &= x & dv &= e^x\,dx \\ du &= dx & v &= e^x \end{aligned}$$

$$\begin{aligned} \int x^2e^x\,dx &= x^2e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] \\ &= x^2e^x - 2xe^x + 2e^x + C \\ &= \boxed{\left(x^2 - 2x + 2\right)e^x + C} \end{aligned}$$

The previous technique works for any integral of the form $\int x^n e^{mx}\,dx$, where n is any *positive* integer and m is any integer. What if n was negative? Then this case we would set $u = e^x$.

Example 3 - Integration by Parts for Definite Integrals: Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



$$A = \int_0^4 xe^{-x}\,dx$$

$$u = x \qquad dv = e^{-x}\,dx$$
$$du = dx \qquad v = -e^{-x}$$

$$\begin{aligned}
\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big|_0^4 - \int_0^4 -e^{-x}\,dx \\
&= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx \\
&= \left(-4e^{-4} - 0\right) - e^{-x}\Big|_0^4 \\
&= -4e^{-4} - \left(e^{-4} - 1\right) \\
&= -4e^{-4} - \left(e^{-4} - 1\right) \\
&= \boxed{-5e^{-4} + 1}
\end{aligned}$$

Example 4 - Tabular Method: In Example 2 we have to apply the Integration by Parts Formula multiple times. There is a convenient way to "book-keep" our work. This is done by creating a table. Let's see how by examining Example 2 again.

Evaluate

$$\int x^2 e^x\,dx.$$

Let $f(x) = x^2$ and $g(x) = e^x$. Then,

Differentiate $f(x)$		Integrate $g(x)$
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

Then the integral is,

$$\begin{aligned}
\int x^2 e^x\,dx &= +x^2 \cdot e^x - 2x \cdot e^x + 2 \cdot e^x + C \\
&= \boxed{\left(x^2 - 2x + 2\right) e^x + C}
\end{aligned}$$

We *have* actually used the integration by parts formula, but we have just made our lives easier by condensing the work into a neat table. This method is extremely useful when Integration by Parts needs to be used over and over again.

Example 5: Find the integral

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos\left(nx\right)\, dx,$$

where n is a positive integer.

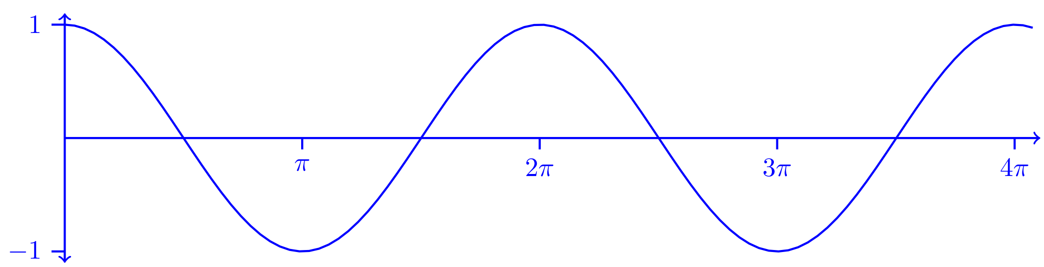
Let $f(x) = x^2$ and $g(x) = \cos(nx)$. Then,

Differentiate $f(x)$		Integrate $g(x)$
x^3	$+$	$\cos(nx)$
$3x^2$	$-$	$\frac{1}{n}\sin(nx)$
$6x$	$+$	$-\frac{1}{n^2}\cos(nx)$
6	$-$	$-\frac{1}{n^3}\sin(nx)$
0		$\frac{1}{n^4}\cos(nx)$

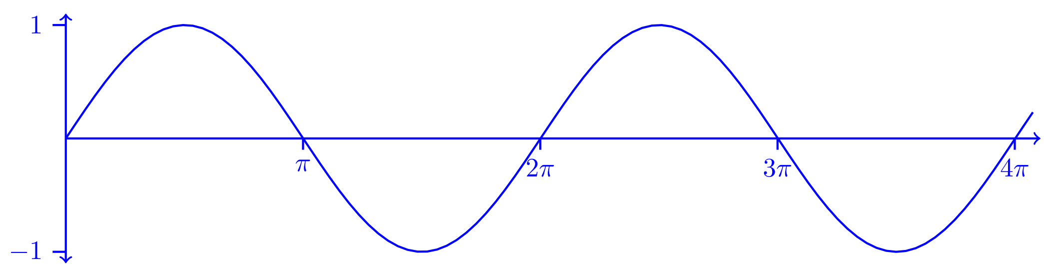
Then the integral is,

$$\begin{aligned}
\frac{1}{\pi}\int x^3\cos(nx)\,dx &= \frac{1}{\pi}\left[+x^3\cdot\frac{1}{n}\sin(nx) - 3x^2\cdot\left(-\frac{1}{n^2}\right)\cos(nx) + 6x\cdot\left(-\frac{1}{n^3}\right)\sin(nx) - 6\cdot\frac{1}{n^4}\cos(nx)\right]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{x^3}{n}\sin(nx) + \frac{3x^2}{n^2}\cos(nx) - \frac{6x}{n^3}\sin(nx) - \frac{6}{n^4}\cos(nx)\right]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\left(0 + \frac{3\pi^2}{n^2}\cos(n\pi) - 0 + \frac{6}{n^4}\right) - \left(0 + 0 - 0 - \frac{6}{n^4}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{3\pi^2(-1)^n}{n^2} - \frac{6(-1)^n}{n^4} + \frac{6}{n^4}\right] \\
&= \boxed{\frac{3}{\pi}\,\frac{\pi^2 n^2(-1)^n - 2(-1)^n + 2}{n^4}}
\end{aligned}$$

cos(x):



sin(x):



Example 6 - Recurring Integrals: Find the integral

$$\int e^x \sin(x)\, dx.$$

We need to apply Integration by Parts twice before we see something:

(1)

$$u = e^x \qquad dv = \sin(x)$$
$$du = e^x\, dx \qquad v = -\cos(x)$$

(2)

$$u = e^x \qquad dv = \cos(x)$$
$$du = e^x\, dx \qquad v = \sin(x)$$

$$\begin{aligned}\int e^x \sin(x)\, dx &= -e^x\cos(x) + \int e^x \cos(x)\, dx \\ &= -e^x\cos(x) + \left(e^x \sin(x) - \int e^x \sin(x)\, dx\right) \\ &= -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\, dx\end{aligned}$$

Notice that now the integral we are interested in, $\int e^x \sin(x)\, dx$, appears on both the left and right hand side of the equation. So, if we add this integral to both sides we get

$$\implies \quad 2\int e^x \sin(x)\, dx = e^x\left(-\cos(x) + \sin(x)\right)$$
$$\implies \quad \int e^x \sin(x)\, dx = \boxed{\frac{e^x\left(\sin(x) - \cos(x)\right)}{2}}$$

This "trick" comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like e^x and trig functions indefinitely and never reach 0. Polynomials on the other hand will eventually "terminate" and their n^{th} derivative (where n is the degree of the polynomial) is identically 0.

Section 8.3: Trigonometric Integrals - Worksheet

Goal: By using trig identities combined with u-substitution, we'd like to find antiderivatives of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

(for integer values of m and n). The goal of this worksheet[1] is for you to work together in groups of 2-3 to discover the techniques that work for these anti-derivatives.

Example 1 - Warm-up: Find

$$\int \cos^4(x)\sin(x)\,dx.$$

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$\begin{aligned}\int \cos^4(x)\sin(x)\,dx &= -\int u^4\,du\\ &= -\frac{u^5}{5}+C\\ &= \boxed{-\frac{\cos^5(x)}{5}+C}\end{aligned}$$

Example 2: Find

$$\int \sin^3(x)\,dx.$$

(Hint: Use the identity $\sin^2(x)+\cos^2(x)=1$, then make a substitution.)

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \left(1-\cos^2(x)\right)\sin(x)\,dx\\ &= -\int \left(1-u^2\right)\,du\\ &= -u+\frac{u^3}{3}+C\\ &= \boxed{-\cos(x)+\frac{\cos^3(x)}{3}+C}\end{aligned}$$

[1]Worksheet adapted from BOALA, `math.colorado.edu/activecalc`

Example 3: Find

$$\int \sin^5(x)\cos^2(x)\,dx.$$

(Hint: Write $\sin^5(x)$ as $\left(\sin^2(x)\right)^2\sin(x)$.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}
\int \sin^5(x)\cos^2(x)\,dx &= \int \left(\sin^2(x)\right)^2\cos^2(x)\sin(x)\,dx \\
&= \int \left(1-\cos^2(x)\right)^2\cos^2(x)\sin(x)\,dx \\
&= -\int \left(1-u^2\right)^2 u^2\,du \\
&= -\int \left(1-2u^2+u^4\right)\,du \\
&= -\int u^2-2u^4+u^6\,du \\
&= -\frac{u^3}{3}+\frac{2u^5}{5}-\frac{u^7}{7}+C \\
&= \boxed{-\frac{\cos^3(x)}{3}+\frac{2\cos^5(x)}{5}-\frac{\cos^7(x)}{7}+C}
\end{aligned}$$

Example 4: Find

$$\int \sin^7(x)\cos^5(x)\,dx.$$

(The algebra here is long. Only set up the substitution - you do not need to fully evaluate.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}
\int \sin^7(x)\cos^5(x)\,dx &= \int \left(\sin^2(x)\right)^3\cos^5(x)\sin(x)\,dx \\
&= \int \left(1-\cos^2(x)\right)^3\cos^5(x)\sin(x)\,dx \\
&= \boxed{-\int \left(1-u^2\right)^3 u^5\,du}
\end{aligned}$$

Example 5: In general, how would you go about trying to find

$$\int \sin^m(x)\cos^n(x)\,dx,$$

where m is odd? (Hint: consider the previous three problems.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}
\int \sin^m(x)\cos^n(x)\,dx &= \int \left(\sin^2(x)\right)^{(m-1)/2}\cos^n(x)\sin(x)\,dx \\
&= \int \left(1-\cos^2(x)\right)^{(m-1)/2}\cos^n(x)\sin(x)\,dx \\
&= \boxed{-\int \left(1-u^2\right)^{(m-1)/2} u^n\,du}
\end{aligned}$$

Example 6: Note that the same kind of trick works when the power on $\cos(x)$ is odd. To check that you understand, what trig identity and what u-substitution would you use to integrate

$$\int \cos^3(x)\sin^2(x)\,dx?$$

$\sin^2(x) + \cos^2(x) = 1$

$$\boxed{\cos^2(x) = 1 - \sin^2(x)}$$

$$\boxed{\begin{aligned} u &= \sin(x) \\ du &= \cos(x)\,dx \end{aligned}}$$

$$\begin{aligned}
\int \cos^3(x)\sin^2(x)\,dx &= \int \cos^2(x)\sin^2(x)\cos(x)\,dx \\
&= \int \left(1 - \sin^2(x)\right)\sin^2(x)\cos(x)\,dx \\
&= \boxed{\int \left(1 - u^2\right)u^2\,du}
\end{aligned}$$

Example 7: Now what if the power on $\cos(x)$ and $\sin(x)$ are both even? Find

$$\int \sin^2(x)\,dx,$$

in each of the following two ways:

(a) Use the identity $\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$.

$$\begin{aligned}
\int \sin^2(x)\,dx &= \int \frac{1}{2}\left(1 - \cos(2x)\right)\,dx \\
&= \frac{1}{2}\int 1 - \cos(2x)\,dx \\
&= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}
\end{aligned}$$

(b) Integrate by parts, with $u = \sin(x)$ and $dv = \sin(x)\,dx$.

$$\begin{aligned}
u &= \sin(x) & dv &= \sin(x)\,dx \\
du &= \cos(x)\,dx & v &= -\cos(x)
\end{aligned}$$

$$\begin{aligned}
\int \sin^2(x)\,dx &= \int \sin(x)\sin(x)\,dx \\
&= -\sin(x)\cos(x) - \int -\cos(x)\cos(x)\,dx \\
&= -\sin(x)\cos(x) + \int \cos^2(x)\,dx \\
&= -\sin(x)\cos(x) + \int 1 - \sin^2(x)\,dx \\
&= -\sin(x)\cos(x) + x - \int \sin^2(x)\,dx
\end{aligned}$$

$$\Longrightarrow 2\int \sin^2(x)\,dx = -\sin(x)\cos(x) + x + C$$

$$\Longrightarrow \int \sin^2(x)\,dx = \boxed{\frac{x - \sin(x)\cos(x)}{2} + C}$$

(c) Show that your answers to parts (a) and (b) above are the same by giving a suitable trig identity.

$$\sin(x)\cos(x) = \frac{1}{2}2\sin(x)\cos(x) = \frac{1}{2}\sin(2x).$$

(d) How would you evaluate the integral

$$\int \sin^2(x)\cos^2(x)\,dx?$$

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}\left(1-\cos(2x)\right)\cdot\frac{1}{2}\left(1+\cos(2x)\right)dx\\
&= \frac{1}{4}\int 1-\cos^2(x)\,dx\\
&= \frac{1}{4}x-\frac{1}{4}\int \cos^2(2x)\,dx\\
&= \frac{1}{4}x-\frac{1}{4}\int \frac{1}{2}\left(1+\cos(4x)\right)dx\\
&= \frac{1}{4}x-\frac{1}{8}x-\frac{1}{8}\int \cos(4x)\,dx\\
&= \boxed{\frac{1}{8}x-\frac{1}{32}\sin(4x)+C}
\end{aligned}$$

Example 8: Evaluate the integral in problem (2) above, again, but this time by parts using $u=\sin^2(x)$ and $dv-\sin(x)\,dx$. (After this, you'll probably need to do a substitution.)

$u=\sin^2(x)$ $\qquad dv=\sin(x)\,dx$

$du=2\sin(x)\cos(x)\,dx$ $\qquad v=-\cos(x)$

$w=\cos(x)$

$dw=-\sin(x)\,dx$

$$\begin{aligned}
\int \sin^3(x)\,dx &= \int \sin^2(x)\sin(x)\,dx\\
&= -\sin^2(x)\cos(x)-\int -\cos(x)\cdot 2\sin(x)\cos(x)\,dx\\
&= -\sin^2(x)\cos(x)+2\int \cos^2(x)\sin(x)\,dx\\
&= -\sin^2(x)\cos(x)-2\int w^2\,dw\\
&= -\sin^2(x)\cos(x)-\frac{2u^3}{3}+C\\
&= \boxed{-\sin^2(x)\cos(x)-\frac{2\cos^3(x)}{3}+C}
\end{aligned}$$

Example 9 - For fun: Can you show your answers to problem (2) and (8) above are the same? It's another great trigonometric identity.

$$-\sin^2(x)\cos(x)-\frac{2\cos^3(x)}{3}=-\left(1-\cos^2(x)\right)\cos(x)-\frac{2}{3}\cos^3(x)=-\cos(x)+\cos^3(x)-\frac{2}{3}\cos^3(x)=-\cos(x)+\frac{\cos^3(x)}{3}$$

Example 10 - Further investigations: (especially for mathematics, physics and engineering majors) We also would like to be able to solve integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

These two functions play well with each other, since the derivative of $\tan(x)$ is $\sec^2(x)$, the derivative of $\sec(x)$ is $\sec(x)|tan(x)$ and since there is a Pythagorean identity relating them. It sometimes works to use $u=\tan(x)$ and it sometimes works to use $u=\sec(x)$. Based on the values of m and n, which substitution should you use? Are there cases for which neither substitution works? (See page 472 of the text.)

Section 8.4: Trigonometric Substitution

Motivation: If we want to find the area of a circle or ellipse, we have an integral of the form

$$\int \sqrt{a^2 - x^2}\, dx$$

where $a > 0$. Regular substitution will not work here, observe:

$$\begin{aligned} u &= a^2 - x^2 \\ du &= -2x\, dx \longleftarrow \text{ extra factor of } x \ldots \end{aligned}$$

Solution: Parametrise! We change x to a function of θ by letting $x = a\sin(\theta)$ so,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin(\theta))^2} = \sqrt{1^2 - a^2\sin^2(\theta)} = \sqrt{a^2\left(1 - \sin^2(\theta)\right)} = \sqrt{a^2\cos^2(\theta)} = a\left|\cos(\theta)\right|.$$

Generally, we use an injective (one-to-one) function (so it has an inverse) to simplify calculations. Above, we ensure $a\sin(\theta)$ is invertible by restricting the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Common Trig Substitutions: The following is a summary of when to use each trig substitution.

Integral contains:	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\left[0, \frac{\pi}{2}\right)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

If you are worried about remembering the identities, then don't! They can all be derived easily, assuming you know three basic ones (which by now you should):

$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \implies \cos^2(\theta) = 1 - \sin^2(\theta) \\ (\div \cos^2(\theta)) \qquad & \tan^2(\theta) + 1 = \sec^2(\theta) \implies \tan^2(\theta) = \sec^2(\theta) - 1 \end{aligned}$$

Example 1: Evaluate

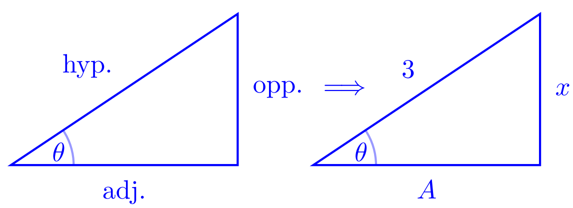
$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$$

$$x = 3\sin(\theta)$$
$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$dx = 3\cos(\theta)\ d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{\sqrt{3^2-3^2\sin^2(\theta)}}{3^2\sin^2(\theta)}\cdot 3\cos(\theta)\ d\theta \\
&= \int \frac{\cancel{3}\sqrt{1-\sin^2(\theta)}}{\cancel{3^2}\sin^2(\theta)}\cdot \cancel{3}\cos(\theta)\ d\theta \\
&= \int \frac{\sqrt{\cos^2(\theta)}}{\sin^2(\theta)}\cdot \cos(\theta)\ d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \int \cot^2(\theta)\ d\theta \\
&= \int \csc^2(\theta) - 1\,d\theta \\
&= -\cot(\theta) - \theta + C \\
&= \boxed{-\frac{\sqrt{3^2-x^2}}{x} - \arcsin(\theta) + C}
\end{aligned}$$

How did we recover x?

$$x = 3\sin(\theta) \Longrightarrow \frac{x}{3} = \sin(\theta)$$



$$A^2 + x^2 = 3^2$$
$$A^2 = 3^2 - x^2$$
$$A = \sqrt{3^2-x^2}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{3^2-x^2}}{x}$$

This is a common process in trig substitution. When you substitute back for your original variable, in this case x, you will always be able to find the correct substitutions by drawing out and labelling a right triangle correctly.

Example 2: Find

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

$x = 2\tan(\theta)$

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$dx = 2\sec^2(\theta)\,d\theta$

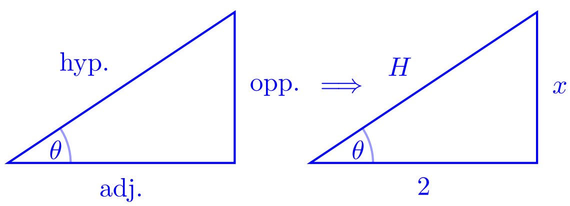
$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{2\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{2^2\tan^2(\theta)+2^2}}\,d\theta \\
&= \int \frac{\cancel{2}\sec^2(\theta)}{2^2\tan^2(\theta)\,\cancel{2}\sqrt{\tan^2(\theta)+1}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec(\theta)}{2^2\tan^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{u^2}\,du \\
&= -\frac{1}{4}\frac{1}{u} + C \\
&= -\frac{1}{4\sin(\theta)} + C \\
&= -\frac{1}{4}\csc(\theta) + C \\
&= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}
\end{aligned}$$

$u = \sin(\theta)$

$du = \cos(\theta)$

How did we recover x?

$x = 2\tan(\theta) \implies \frac{x}{2} = \tan(\theta)$



$H^2 = x^2 + 2^2$

$H = \sqrt{x^2+4}$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hyp.}}{\text{opp.}} = \frac{\sqrt{x^2+4}}{x}$$

Example 3: Evaluate

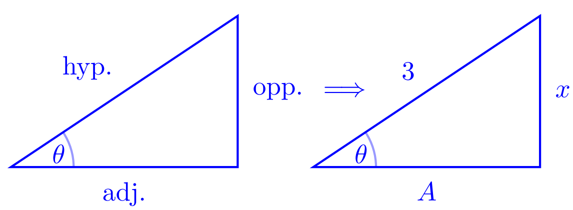
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

$$x = 3\sin(\theta)$$
$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$dx = 3\cos(\theta)\ d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{3^2\sin^2(\theta)}{\sqrt{3^2-3^2\sin^2(\theta)}} \cdot 3\cos(\theta)\ d\theta \\
&= \int \frac{3^2\sin^2(\theta)}{\cancel{3}\sqrt{1-\sin^2(\theta)}} \cdot \cancel{3}\cos(\theta)\ d\theta \\
&= \int \frac{3^2\sin^2(\theta)}{\sqrt{\cos^2(\theta)}} \cdot \cos(\theta)\ d\theta \\
&= 9\int \sin^2(\theta) \\
&= \frac{9}{2}\int 1-\cos(2\theta)\ d\theta \\
&= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{9}{2}\left(\theta - \sin(\theta)\cos(\theta)\right) + C \\
&= \boxed{\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C}
\end{aligned}$$

How did we recover x?

$$x = 3\sin(\theta) \implies \frac{x}{3} = \sin(\theta)$$



$$3^2 = x^2 + A^2$$
$$A = \sqrt{9-x^2}$$

$$\cos(\theta) = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{9-x^2}}{x}$$

Section 8.5: Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that *can* be integrated using methods with which we are already familiar. That is, we cannot integrate

$$\frac{1}{x^2 - x}$$

as-is, but it is equivalent to

$$\frac{1}{x} - \frac{1}{x-1},$$

each term of which we can integrate.

Example 1: Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)}\, dx.$$

(a) $\displaystyle\int \frac{1}{x+1}\, dx = \ln|x+1| + C$

(b) $\displaystyle\frac{2}{x+1} - \frac{1}{x-3} = \frac{2(x-3)-(x+1)}{(x+1)(x-3)} = \frac{2x-6-x-1}{(x+1)(x-3)} = \frac{x-7}{(x+1)(x-3)}$

(c) $\displaystyle\int \frac{x-7}{(x+1)(x-3)}\, dx = \int \frac{2}{x+1} - \frac{1}{x-3}\, dx = 2\ln|x+1| - \ln|x-3| + C$

Example 2: Compute $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx$.

(a) $\displaystyle\frac{7}{x-1} + \frac{3}{x-4} = \frac{7(x-4)+3(x-1)}{(x-1)(x-4)} = \frac{10x-31}{(x-1)(x-4)}$

(b) $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx = \int \frac{7}{x-1} + \frac{3}{x-4}\, dx = 7\ln|x-1| + 3\ln|x-4| + C$

The previous two examples were nice since we were given a different expression of our integrand before hand. But what about when we don't? It is clear that the key step is decomposing our integrand into simple pieces, so how do we do it? The next example outlines the method.

Example 3: Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Our first step is to decompose $\frac{x+14}{(x+5)(x+2)}$ as

$$\frac{x+14}{(x+5)(x+2)} = \frac{?}{x+5} + \frac{?}{x+2}.$$

There is no indicator of what the numerators should be, so there is work to be done to find them. If we let the numerators be variables, we can use algebra to solve. That is, we want to find constants A and B that make the equation below true for all $x \neq -5, -2$.

$$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}.$$

We solve for A and B by cross multiplying and equating the numerators.

$$\begin{aligned}\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} = \frac{A(x+2)+B(x+5)}{(x+5)(x+2)} \Longrightarrow x+14 &= A(x+2)+B(x+5)\\ &= Ax+2A+Bx+5B\\ &= (A+B)x+2A+5B\end{aligned}$$

$$\begin{aligned}1 &= A+B \Longrightarrow B = 1-A\\ 14 &= 2A+5B\\ &= 2A+5(1-A)\\ &= 2A+5-5A\\ &= 5-3A\\ \Longrightarrow 9 &= -3A\\ \Longrightarrow -3 &= A\\ \Longrightarrow B &= 1-(-3) = 4\end{aligned}$$

$$\begin{aligned}\int \frac{x+14}{(x+5)(x+2)}\,dx &= \int \frac{-3}{x+5} + \frac{4}{x+2}\,dx\\ &= \boxed{-3\ln|x+5| + 4\ln|x+2| + C}\end{aligned}$$

Example 4: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

$$\begin{aligned}\frac{x+15}{(3x-4)(x+1)} = \frac{A}{3x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(3x-4)}{(3x-4)(x+1)} \Longrightarrow x+15 &= A(x+1)+B(3x-4)\\ &= Ax+A+3Bx-4B\\ &= (A+3B)x+A-4B\end{aligned}$$

$$\begin{aligned}1 &= A+3B \Longrightarrow A = 1-3B\\ 15 &= A-4B\\ &= (1-3B)-4B\\ &= 1-7B\\ \Longrightarrow 14 &= -7B\\ \Longrightarrow -2 &= B\\ \Longrightarrow A &= 1-3(-2) = 7\end{aligned}$$

$$\begin{aligned}\int \frac{x+15}{(3x-4)(x+1)}\,dx &= \int \frac{7}{3x-4} - \frac{2}{x+1}\,dx\\ &= \boxed{\frac{7}{3}\ln|3x+5| - 2\ln|x+1| + C}\end{aligned}$$

Example 4 - An alternative approach: Find

$$\int \frac{x+15}{(3x-4)\,(x+1)}\,dx.$$

$$\frac{x+15}{(3x-4)\,(x+1)} = \frac{A}{3x-4} + \frac{B}{x+1} = \frac{A\,(x+1)+B\,(3x-4)}{(3x-4)\,(x+1)} \Longrightarrow x+15 = A\,(x+1)+B\,(3x-4)$$

Instead of expanding everything, comparing coefficients and solving a system of linear equations, sometimes it may be helpful to plug in strategic values of x to solve. Good values to choose are those that are roots of the polynomials that appear on the denominators of the fraction. Observe,

$$\begin{aligned}
x=-1: &\quad (-1)+15 = A\,((-1)+1)+B\,(3(-1)-4)\\
\Longrightarrow &\quad 14 = 0-7B\\
\Longrightarrow &\quad -2 = B
\end{aligned}$$

$$\begin{aligned}
x=\tfrac{4}{3}: &\quad \left(\tfrac{4}{3}\right)+15 = A\left(\left(\tfrac{4}{3}\right)+1\right)+B\left(3\left(\tfrac{4}{3}\right)-4\right)\\
\Longrightarrow &\quad \tfrac{49}{3} = \tfrac{7}{3}A+0\\
\Longrightarrow &\quad 7 = A
\end{aligned}$$

$$\int \frac{x+15}{(3x-4)\,(x+1)}\,dx = \int \frac{7}{3x-4} - \frac{2}{x+1}\,dx = \boxed{\frac{7}{3}\ln|3x+5| - 2\ln|x+1| + C}$$

Example 5: Goal: Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} = \frac{A+B}{x+3}.$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2} \Longrightarrow 5x-2 = A(x+3)+B$$

$$\begin{aligned}
x=-3: &\quad 5(-3)-2 = A\,((-3)+3)+B\\
\Longrightarrow &\quad -17 = 0+B\\
\Longrightarrow &\quad -17 = B
\end{aligned}$$

$$\begin{aligned}
5x-2 &= A(x+3)-17\\
&= Ax+3A-17\\
\Longrightarrow \quad 5x &= Ax\\
\Longrightarrow \quad 5 &= A
\end{aligned}$$

$$\int \frac{5x-2}{(x+3)^2}\,dx = \int \frac{5}{x+3} - \frac{17}{(x+3)^2}\,dx = \boxed{5\ln|x+3| + \frac{17}{x+3} + C}$$

Example 6: What if the denominator is an irreducible quadratic of the form x^2+px+q? That is, it can not be factored (does not have any real roots). In this case, suppose that $\left(x^2+px+q\right)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)}+\frac{B_1x+C_1}{(x^2+px+q)^2}+\frac{B_1x+C_1}{(x^2+px+q)^3}+\cdots+\frac{B_1x+C_1}{(x^2+px+q)^n}.$$

Compute $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

$$\frac{-2x+4}{(x^2+1)(x-1)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{(x-1)^2}=\frac{(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)+D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$\Longrightarrow -2x+4=(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)+D(x^2+1)$$

There are four unknowns here, A, B, C and D. In this case we're going to want to minimise the amount of work we do here. In general it is going to be beneficial to solve for as many coefficients as we can by plugging in numbers, and then expand everything to compare coefficient after reducing the workload.

$$\begin{aligned}
x=1: &\quad -2(1)+4=(Ax+B)((1)-1)^2+C((1)^2+1)((1)-1)+D((1)^2+1)\\
\Longrightarrow &\quad 2=0+0+2D\\
\Longrightarrow &\quad 1=D
\end{aligned}$$

So we got one coefficient this way. That's better than nothing! Now if we use this new information and then rearrange a little we end up with less solving to do. This *does* require you however to be comfortable with algebra.

$$\begin{aligned}
\Longrightarrow &\quad -2x+4=(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)+(x^2+1)\\
\Longrightarrow &\quad -x^2-2x+3=(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)\\
&\quad -(x^2+2x-3)=\\
&\quad -(x-1)(x+3)=
\end{aligned}$$

Now we have already seen what happens when $x=1$, so we can go right ahead and divide by the $(x-1)$ term that appears on both sides.

$$\begin{aligned}
\Longrightarrow \quad -x-3&=(Ax+B)\,(x-1)+C(x^2+1)\\
&=Ax^2+Bx-Ax-B+Cx^2+C\\
&=(A+C)x^2+(B-A)x+C-B
\end{aligned}$$

Now we can go through and set up equations and solve by coefficients. When there are lots of coefficients it is a good idea of coming up with a way to book-keep your algebra - it can get *very* messy if you don't. Below is just one way you can do it.

$$\begin{pmatrix}(1) & 0 & = & A & & +C\\ (2) & -1 & = & -A & +B & \\ (3) & -3 & = & & -B & +C\end{pmatrix}\overset{(2)+(3)}{\Longrightarrow}\begin{pmatrix}(1) & 0 & = & A & & +C\\ (2) & -4 & = & -A & & +C\\ (3) & -3 & = & & -B & +C\end{pmatrix}\overset{(1)+(2)}{\Longrightarrow}\begin{pmatrix}(1) & -4 & = & & & 2C\\ (2) & -4 & = & -A & & +C\\ (3) & -3 & = & & -B & +C\end{pmatrix}$$

$$\begin{aligned}
&\Longrightarrow\ -4=2C &&\Longrightarrow\ -2=C\\
&\Longrightarrow\ -4=-A-2 &&\Longrightarrow\ 2=A\\
&\Longrightarrow\ -3=-B-2 &&\Longrightarrow\ 1=B,\ \text{So}\ldots
\end{aligned}$$

$$\begin{aligned}
\int\frac{-2x+4}{(x^2+1)(x-1)^2}\,dx&=\int\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\,dx\\
&=\int\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\,dx\\
&=\boxed{\ln(x^2+1)+\tan^{-1}(x)-2\ln|x-1|-\frac{1}{x-1}+C}
\end{aligned}$$

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper ($\deg f(x) < \deg g(x)$)

1. Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(X)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_1x+C_1}{(x^2+px+q)^2} + \frac{B_1x+C_1}{(x^2+px+q)^3} + \cdots + \frac{B_1x+C_1}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are
   (i) One fraction for each power of the irreducible factor that appears
   (ii) The degree of the numerator should be one less than the degree of the denominator

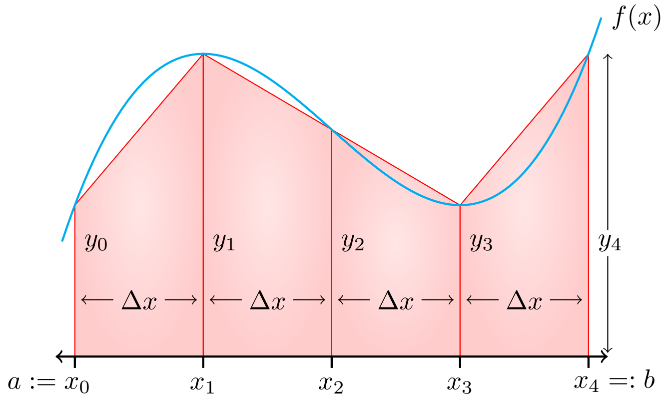
4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

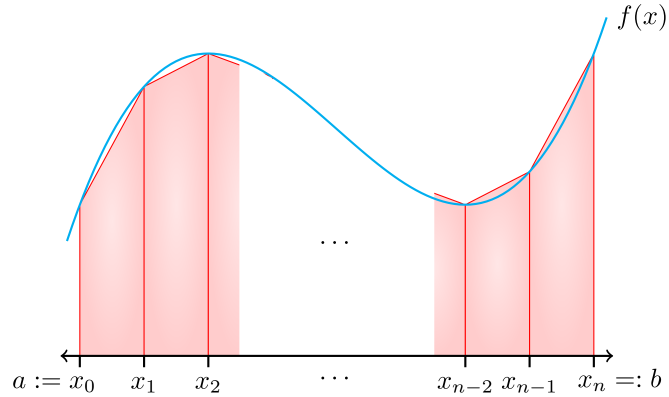
5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.

Section 8.7: Numerical Integration

What to do when there's no nice antiderivative? The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln(x)$ and $\sqrt{1+x^4}$ have no elementary formulas/ When we cannot find a workable antiderivative for a function $f(x)$ that we have to integrate, we can partition the interval of integration, replace $f(x)$ by a closely fitting polynomial on each subinterval, integrate the poynomials and add the results to *approximate* the definite integral of $f(x)$. This is an example of numerical integration. There are many methods of numerical integration but we will study only two: the *Trapezium Rule* and *Simpson's Rule*.

Trapezoidal Approximations: As the name implies, the Trapezium Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapeziums instead of rectangles - which, if you recall, we studied when we looked at Riemann integration in Calculus I.





Assume the length of each subinterval is $\Delta x = \dfrac{b-a}{n}$. Then the area of the trapezium that lies above the x-axis in the i^{th} subinterval is $T_i = \dfrac{\delta x}{2}\left(y_{i-1} + y_i\right)$ where $y_{i-1} = f\left(x_{i-1}\right)$ and $y_i = f\left(x_i\right)$. Then the area of the under the curve and above the x-axis is approximated by the sum of the trapeziums:

$$\begin{aligned}
T &= \frac{\Delta x}{2}\left(y_0 + y_1\right) + \frac{\Delta x}{2}\left(y_1 + y_2\right) + \cdots + \frac{\Delta x}{2}\left(y_{n-1} + y_n\right) \\
&= \frac{\Delta x}{2}\left(y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n\right) \\
&= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \\
&= \frac{\Delta x}{2}\left(y_0 + y_n + 2\sum_{i=1}^{n-1} y_i\right) \\
&= \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f\left(x_i\right)\right)
\end{aligned}$$

The Trapezium Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right), \end{aligned}$$

where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$.

Example 1: Use the Trapezium Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Partition the interval $[1, 2]$ into 4 subintervals:

$$\begin{aligned} \Delta x &= \frac{2-1}{4} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} x_0 &= a \\ &= 1 \\ &= \frac{4}{4} \end{aligned} \qquad \begin{aligned} x_1 &= a + \Delta x \\ &= 1 + 1\cdot\frac{1}{4} \\ &= \frac{5}{4} \end{aligned} \qquad \begin{aligned} x_2 &= a + 2\Delta x \\ &= 1 + 2\cdot\frac{1}{4} \\ &= \frac{6}{4} \end{aligned} \qquad \begin{aligned} x_3 &= a + 3\Delta x \\ &= 1 + 3\cdot\frac{1}{4} \\ &= \frac{7}{4} \end{aligned} \qquad \begin{aligned} x_4 &= a + 4\Delta x \\ &= 1 + 4\cdot\frac{1}{4} \\ &= \frac{8}{4} \end{aligned}$$

Now use these points together with the formula for the Trapezium Rule:

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1/4}{2}\left(f\left(\frac{4}{4}\right) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 2f\left(\frac{7}{4}\right) + f\left(\frac{8}{4}\right)\right) \\ &= \frac{1}{8}\left(\frac{16}{16} + 2\frac{25}{16} + 2\frac{36}{16} + 2\frac{49}{16} + \frac{64}{16}\right) \\ &= \frac{1}{128}(16 + 50 + 72 + 98 + 64) \\ &= \frac{1}{128}(300) \\ &= \boxed{\frac{75}{32}} \end{aligned}$$

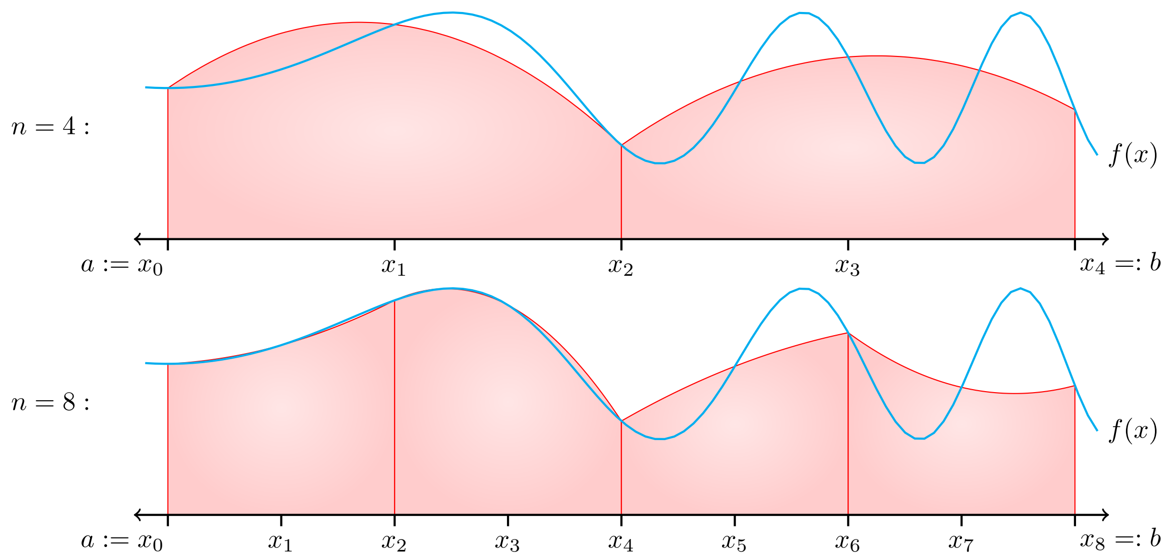
$$\begin{aligned} \int_1^2 x^2\,dx &= \frac{1}{3}x^3\Big|_1^2 \\ &= \frac{1}{3}\left(2^3 - 1^3\right) \\ &= \frac{1}{3}(8-1) \\ &= \boxed{\frac{7}{3}} \end{aligned}$$

$$\frac{75}{32} - \frac{7}{3} = \frac{225}{96} - \frac{224}{96} = \frac{1}{96}.$$

So the approximation overestimated the actual area by $\frac{1}{96}$, which is pretty good considering we only used 4 trapeziums.

Just like when we looked at Riemann sums, using more trapeziums results in a better approximation.

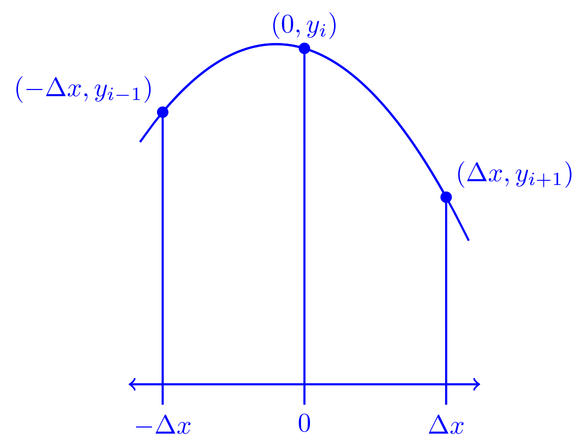
Parabolic Approximations: Instead of using the straight-line segments that produced the trapeziums, we can use parabolas to approximate the definite integral of a continuous function. We partition the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$ but this time we require n to be an even number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola. A typical parabola passed through three consecutive points: (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) on the curve.



So how do we compute the area under each parabola $y = Ax^2 + Bx + C$? By translating we can assume that the centre point of our parabola is at $x_i = 0$

The area under the parabola and above the x-axis is given by

$$
\begin{aligned}
S_i &= \int_{-\Delta x}^{\Delta x} Ax^2 + Bx + C \, dx \\
&= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big|_{-\Delta x}^{\Delta x} \\
&= \frac{A(\Delta x)^3}{3} + \frac{B(\Delta x)^2}{2} + C(\Delta x) - \left[\frac{A(-\Delta x)^3}{3} + \frac{B(-\Delta x)^2}{2} + C(-\Delta x)\right] \\
&= \frac{2A\Delta x^3}{3} + 2C\Delta x \\
&= \frac{\Delta x}{3}\left(2A\Delta x + 6C\right)
\end{aligned}
$$



$$
\left.\begin{aligned}
y_{i-1} &= A\Delta x^2 - B\Delta x + C \\
y_i &= C \\
y_{i+1} &= A\Delta x^2 + B\Delta x + C
\end{aligned}\right\} \implies y_{i-1} + 4y_i + y_{i+1} = \left(A\Delta x^2 - B\Delta x + C\right) + 4C + \left(A\Delta x^2 + B\Delta x + C\right) = 2A\Delta x + 6C
$$

$$
\implies \boxed{S_i = \frac{\Delta x}{3}\left(y_{i-1} + 4y_i + y_{i+1}\right)}
$$

So if we sum up the areas under all of the parabolas, we obtain our approximation.

Simpson's Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{3}\left(f(x_0) + f(x_n) + 2\left(\sum_{i=1}^{\frac{n-1}{2}} f(x_{2i-1}) + 2f(x_{2i})\right)\right), \end{aligned}$$

where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$ with n an *even* number.

Example 2: Use the Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$. Compare the estimate with the exact value.

Partition the interval $[1, 2]$ into 4 subintervals:

$$\begin{aligned} \Delta x &= \frac{2-0}{4} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} x_0 &= a \\ &= 0 \\ &= \frac{0}{2} \end{aligned} \qquad \begin{aligned} x_1 &= a + \Delta x \\ &= 0 + 1 \cdot \frac{1}{2} \\ &= \frac{1}{2} \end{aligned} \qquad \begin{aligned} x_2 &= a + 2\Delta x \\ &= 0 + 2 \cdot \frac{1}{2} \\ &= \frac{2}{2} \end{aligned} \qquad \begin{aligned} x_3 &= a + 3\Delta x \\ &= 0 + 3 \cdot \frac{1}{2} \\ &= \frac{3}{2} \end{aligned} \qquad \begin{aligned} x_4 &= a + 4\Delta x \\ &= 0 + 4 \cdot \frac{1}{2} \\ &= \frac{4}{2} \end{aligned}$$

Now use these points together with the formula for Simpson's Rule:

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1/2}{3}\left(f\left(\frac{0}{2}\right) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right)\right) \\ &= \frac{1}{6}\left(5\frac{0}{16} + 4 \cdot 5\frac{1}{16} + 2 \cdot 5\frac{16}{16} + 4 \cdot 5\frac{81}{16} + 5\frac{128}{16}\right) \\ &= \frac{5}{96}\left(0 + 4 + 32 + 324 + 256\right) \\ &= \frac{5}{96}(616) \\ &= \boxed{\frac{385}{12}} \end{aligned}$$

$$\begin{aligned} \int_0^2 5x^4\,dx &= x^5\Big|_0^2 \\ &= 2^5 - 0^5 \\ &= 32 - 0 \\ &= \boxed{32} \end{aligned}$$

$$\frac{385}{12} - 32 = \frac{385}{12} - \frac{384}{12} = \frac{1}{12}.$$

So the approximation overestimated the actual area by $\frac{1}{12}$, which is pretty good considering we only used 2 parabolas.

Just like Riemann sums and the Trapezium rule, using more parabolas results in a better approximation. In fact, of the three rules Simpson's Rule gives the best approximation. This can be seen by looking at the *error estimates*.

Error Estimates in the Trapezium and Simpson's Rules If $f''(x)$ is continuous and M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, then the error E_T in the Trapezium Rule for approximating the definite integral of $f(x)$ over the interval $[a, b]$ using n trapeziums satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

If $f^{(4)}(x)$ is continuous and M is any upper bound for the values of $|f^{(4)}(x)|$ on $[a, b]$, then the error E_S in Simpson's Rule for approximating the definite integral of $f(x)$ over the interval $[a, b]$ using $\frac{n}{2}$ parabolas satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

Example 3: Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$. What value of n should we pick so that the error is within 0.001 of the true value?

First we differentiate $f(x)$ 4 times and check that it is continuous on the interval $[0, 2]$.

$$\begin{aligned} f(x) &= 5x^4 \\ f'(x) &= 20x^3 \\ f''(x) &= 60x^2 \\ f'''(x) &= 120x \\ f^{(4)}(x) &= 120 \end{aligned}$$

This is a constant function, so it *is* continuous on our interval. Further

$$|f^{(4)}(x)| = 120 \leq 120 \text{ for all } x \in [0, 2].$$

Thus $M = 120$ works as a bound. So, with $n = 4$, the error is bounded by:

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2-0)^5}{180(4)^4} = \frac{120 \cdot 2^5}{180 \cdot 2^8} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}.$$

To achieve an approximation with $|E_S| \leq 0.001$, we again find a bound for M but this time we solve the inequality for n.

$$\frac{M(b-a)^5}{180n^4} = \frac{120(2-0)^5}{180n^4} = \frac{2^6}{3n^4} \leq 0.001$$

$$\begin{aligned} \Longrightarrow \quad & \frac{2^6}{3} \leq \frac{1}{1000} n^4 \\ \Longrightarrow \quad & \frac{2^6 \cdot 1000}{3} \leq n^4 \\ \Longrightarrow \quad & \frac{2^8 \cdot 2 \cdot 5^3}{3} \leq n^4 \\ \Longrightarrow \quad & 4\sqrt[4]{\frac{2 \cdot 5^3}{3}} \leq n \end{aligned}$$

So setting $n \geq 4\sqrt[4]{\frac{2 \cdot 5^3}{3}} \approx 12.086$ would ensure an approximation of the desired accuracy.

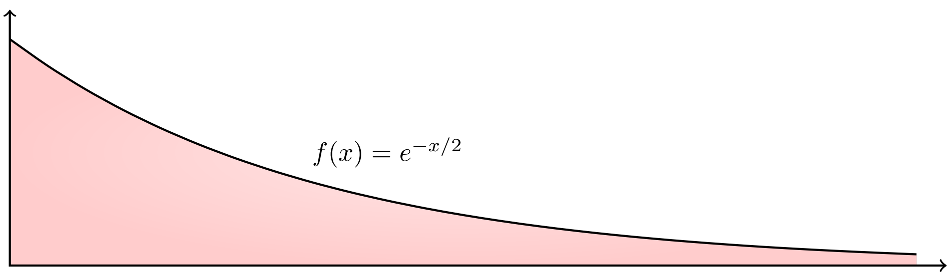
Section 8.8: Improper Integrals

Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite
2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



First, we examine what the area looks like over finite intervals. That is, we integrate over $[0, b]$.

$$A(b) := \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} - \left[-2e^{-0/2}\right] = 2\left(1 - e^{-b/2}\right).$$

Now we have an expression for the area over a finite integral, we can let $b \longrightarrow \infty$ by calculating the limit of this expression.

$$A = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2\left(1 - e^{-b/2}\right) = 2\,(1-0) = 2.$$

So,

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

So this is how we deal with infinite limits of integration - with a limit! Remember those?

Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty} \int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we sat that the improper integral ___converges___ and that the limit is the ___value___ of the improper integral. If the limit fails to exist, the improper integral ___diverges___

Any of the integrals in the above definition can be interpreted as an area if $f(x) \geq 0$ on the interval of integration. If $f(x) \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

$$u = \ln(x) \qquad dv = \frac{1}{x^2}\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$$

$$\begin{aligned}\int_1^b \frac{\ln(x)}{x^2}\,dx &= -\frac{\ln(x)}{x}\bigg|_1^b - \int_1^b -\frac{1}{x^2}\,dx \\ &= -\frac{\ln(x)}{x} - \frac{1}{x}\bigg|_1^b \\ &= -\frac{\ln(b)}{b} - \frac{1}{b} - \left[-\frac{\ln(1)}{1} - \frac{1}{1}\right] \\ &= -\frac{\ln(b)}{b} - \frac{1}{b} + 1\end{aligned}$$

Now we take a limit,

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b} - \frac{1}{b} + 1\right] = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b}\right] - 0 + 1 \overset{\text{L'H}}{=} \lim_{b\to\infty}\left[-\frac{1/b}{1}\right] + 1 = 0 + 1 = \boxed{1}$$

L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that $f(x)$ and $g(x)$ are differentiable on an open interval I containing a and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the left and right both exist.

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

According to part 3 of our definition, we can choose any real number c and split this integral into two integrals and then apply parts 1 and 2 to each piece. Let's choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Now we will evaluate each piece separately.

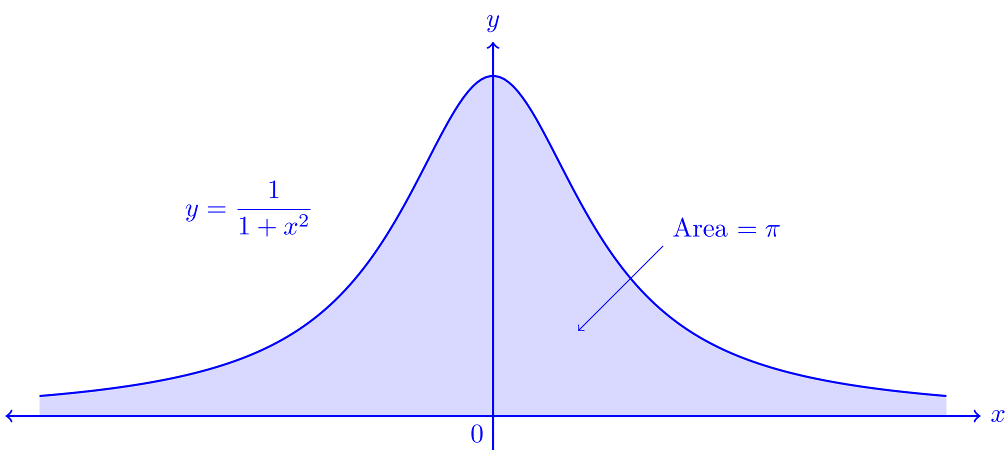
$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx \\ &= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_{1}^{0} \\ &= \lim_{a\to-\infty} \tan^{-1}(0) - \tan^{-1}(a) \\ &= \lim_{a\to-\infty} -\tan^{-1}(a) \\ &= \frac{\pi}{2},\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \tan^{-1}(x)\Big|_{0}^{b} \\ &= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(0) \\ &= \lim_{b\to\infty} \tan^{-1}(b) \\ &= \frac{\pi}{2}.\end{aligned}$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Since $1/(1+x^2) > 0$ on $\mathbb{R}$, the improper integral can be interpreted as the (finite) area between the curve and the x-axis.



A Special Example: For what values of p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

We split this investigation into two cases; when $p \neq 1$ and when $p = 1$.

If $p \neq 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^b \\ &= \lim_{b\to\infty} \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\bigg|_1^b \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p<1.\end{cases}\end{aligned}$$

If $p = 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln(x)\bigg|_1^b \\ &= \lim_{b\to\infty}[\ln(b)-\ln(1)] \\ &= \lim_{b\to\infty}\ln(b) = \infty\end{aligned}$$

Combining these two results we have

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p\leq 1\end{cases}}$$

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

First we find the integral over the region $[a,1]$ where $0 < a \leq 1$.

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-1/2}\,dx = 2x^{1/2}\bigg|_a^1 = 2\sqrt{x}\bigg|_a^1 = 2-2\sqrt{a} = 2(1-\sqrt{a}).$$

Then we find the limit as $a \to 0^+$:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\left(1-\sqrt{a}\right) = 2.$$

Therefore,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^a f(x)\,dx.$$
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral ____converges____ and that the limit is the ____value____ of the improper integral. If the limit fails to exist, the improper integral ____diverges____

Example 5: Investigate the convergence of
$$\int_0^1 \frac{1}{1-x}\,dx.$$

$$\begin{aligned}
\int_0^1 \frac{1}{1-x}\,dx &= \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx \\
&= \lim_{b\to 1^-} -\int_0^b \frac{1}{x-1}\,dx \\
&= \lim_{b\to 1^-} -\ln|x-1|\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(x-1)\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(1-b) \\
&= -(-\infty) \\
&= \boxed{\infty}
\end{aligned}$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Why does this make sense?

1. If the area under the curve of $g(x)$ is *finite* and $f(x)$ is bounded above by $g(x)$ (and below by 0), then the area under the curve of $f(x)$ must be *less than or equal to* the area under the curve of $g(x)$. A positive number less that a *finite* number is also *finite.*
2. If the area under the curve of $f(x)$ is *infinite* and $g(x)$ is bounded below by $f(x)$, then the area under the curve of $g(x)$ must be *"less than or equal to"* the area under the curve of $g(x)$. Since there is no finite number "greater than" infinity, the area under $g(x)$ must also be *infinite.*

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

We want to find a function $g(x)$ such that for some $a \in \mathbb{R}$, $f(x) = \dfrac{\cos^2(x)}{x^2} \leq g(x)$ or $f(x) = \dfrac{\cos^2(x)}{x^2} \geq g(x)$ for all $x \geq a$. One way we can do this is by finding *bounds* for $f(x)$. Since $0 \leq \cos^2(x) \leq 1$ for all x,

$$\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

So then we can use $g(x) := \dfrac{1}{x^2}$. So,

$$0 \leq \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx \leq \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{b} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

So $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Since $x \geq x - e^{-x}$, $f(x) := \dfrac{1}{x} \leq \dfrac{1}{x - e^{-1}} =: g(x)$ for all $x \geq 3$. So,

$$0 \leq \int_3^\infty f(x)\,dx \leq \int_3^\infty g(x)\,dx.$$

By the Direct Comparison Test then, $\displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,dx$ *diverges* since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges.

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Why does this make sense? The convergence is really only dependent on the "tail" of the integral. That is, the convergence is dictated by what happens "at infinity." If for sufficiently large values of x, $f(x) \approx Lg(x)$ and one of the two integrals converges, then the other one should also converge, since it is only off by "about a scalar multiple." The same goes for diverging, if one diverges, then multiplying it by a positive number won't suddenly make it converge, so the other one should also diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Let $f(x) := \dfrac{1}{1+x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1+x^2-1}{1+x^2} = \lim_{x\to\infty} \left(1 - \frac{1}{1+x^2}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ *converges.*

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.

Let $f(x) := \dfrac{1-e^{-x}}{x}$ and $g(x) := \dfrac{1}{x}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(1 - e^{-x}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ *diverges.*