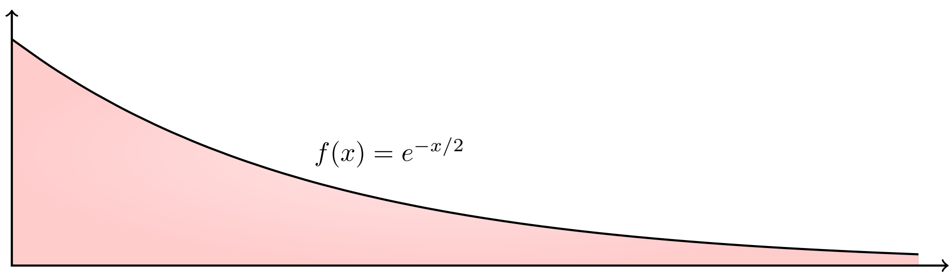
Section 8.8: Improper Integrals

Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite
2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty} \int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we sat that the improper integral ____________________ and that the limit is the

____________________ of the improper integral. If the limit fails to exist, the improper integral ____________________

Any of the integrals in the above definition can be interpreted as an area if $f(x) \geq 0$ on the interval of integration. If $f(x) \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

A Special Example: For what values of p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^a f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral ______________________ and that the limit is the

___________________ of the improper integral. If the limit fails to exist, the improper integral ______________________

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$.
2. If $\int_a^\infty f(x)\,dx$ diverges, then so does $\int_a^\infty g(x)\,dx$.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.