Section 8.5: Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that *can* be integrated using methods with which we are already familiar. That is, we cannot integrate

$$\frac{1}{x^2 - x}$$

as-is, but it is equivalent to

$$\frac{1}{x} - \frac{1}{x-1},$$

each term of which we can integrate.

Example 1: Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)}\,dx.$$

(a) $\displaystyle\int \frac{1}{x+1}\,dx = \ln|x+1| + C$

(b) $\displaystyle\frac{2}{x+1} - \frac{1}{x-3} = \frac{2(x-3)-(x+1)}{(x+1)(x-3)} = \frac{2x-6-x-1}{(x+1)(x-3)} = \frac{x-7}{(x+1)(x-3)}$

(c) $\displaystyle\int \frac{x-7}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1} - \frac{1}{x-3}\,dx = 2\ln|x+1| - \ln|x-3| + C$

Example 2: Compute $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\,dx$.

(a) $\displaystyle\frac{7}{x-1} + \frac{3}{x-4} = \frac{7(x-4)+3(x-1)}{(x-1)(x-4)} = \frac{10x-31}{(x-1)(x-4)}$

(b) $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\,dx = \int \frac{7}{x-1} + \frac{3}{x-4}\,dx = 7\ln|x-1| + 3\ln|x-4| + C$

The previous two examples were nice since we were given a different expression of our integrand before hand. But what about when we don't? It is clear that the key step is decomposing our integrand into simple pieces, so how do we do it? The next example outlines the method.

Example 3: Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Our first step is to decompose $\frac{x+14}{(x+5)(x+2)}$ as

$$\frac{x+14}{(x+5)(x+2)} = \frac{?}{x+5} + \frac{?}{x+2}.$$

There is no indicator of what the numerators should be, so there is work to be done to find them. If we let the numerators be variables, we can use algebra to solve. That is, we want to find constants A and B that make the equation below true for all $x \neq -5, -2$.

$$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}.$$

We solve for A and B by cross multiplying and equating the numerators.

$$\begin{aligned}\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} = \frac{A(x+2)+B(x+5)}{(x+5)(x+2)} \Longrightarrow x+14 &= A(x+2)+B(x+5)\\ &= Ax+2A+Bx+5B\\ &= (A+B)x+2A+5B\end{aligned}$$

$$\begin{aligned}1 &= A+B \Longrightarrow B = 1-A\\ 14 &= 2A+5B\\ &= 2A+5(1-A)\\ &= 2A+5-5A\\ &= 5-3A\\ \Longrightarrow 9 &= -3A\\ \Longrightarrow -3 &= A\\ \Longrightarrow B &= 1-(-3) = 4\end{aligned}$$

$$\begin{aligned}\int \frac{x+14}{(x+5)(x+2)}\,dx &= \int \frac{-3}{x+5} + \frac{4}{x+2}\,dx\\ &= \boxed{-3\ln|x+5| + 4\ln|x+2| + C}\end{aligned}$$

Example 4: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

$$\begin{aligned}\frac{x+15}{(3x-4)(x+1)} = \frac{A}{3x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(3x-4)}{(3x-4)(x+1)} \Longrightarrow x+15 &= A(x+1)+B(3x-4)\\ &= Ax+A+3Bx-4B\\ &= (A+3B)x+A-4B\end{aligned}$$

$$\begin{aligned}1 &= A+3B \Longrightarrow A = 1-3B\\ 15 &= A-4B\\ &= (1-3B)-4B\\ &= 1-7B\\ \Longrightarrow 14 &= -7B\\ \Longrightarrow -2 &= B\\ \Longrightarrow A &= 1-3(-2) = 7\end{aligned}$$

$$\begin{aligned}\int \frac{x+15}{(3x-4)(x+1)}\,dx &= \int \frac{7}{3x-4} - \frac{2}{x+1}\,dx\\ &= \boxed{\frac{7}{3}\ln|3x+5| - 2\ln|x+1| + C}\end{aligned}$$

Example 4 - An alternative approach: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

$$\frac{x+15}{(3x-4)(x+1)} = \frac{A}{3x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(3x-4)}{(3x-4)(x+1)} \Longrightarrow x+15 = A(x+1)+B(3x-4)$$

Instead of expanding everything, comparing coefficients and solving a system of linear equations, sometimes it may be helpful to plug in strategic values of x to solve. Good values to choose are those that are roots of the polynomials that appear on the denominators of the fraction. Observe,

$$\begin{aligned} x=-1: &\quad (-1)+15 = A((-1)+1)+B(3(-1)-4) \\ \Longrightarrow &\quad 14 = 0-7B \\ \Longrightarrow &\quad -2 = B \end{aligned}$$

$$\begin{aligned} x=\tfrac{4}{3}: &\quad \left(\tfrac{4}{3}\right)+15 = A\left(\left(\tfrac{4}{3}\right)+1\right)+B\left(3\left(\tfrac{4}{3}\right)-4\right) \\ \Longrightarrow &\quad \tfrac{49}{3} = \tfrac{7}{3}A+0 \\ \Longrightarrow &\quad 7 = A \end{aligned}$$

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx = \int \frac{7}{3x-4} - \frac{2}{x+1}\,dx = \boxed{\frac{7}{3}\ln|3x+5| - 2\ln|x+1| + C}$$

Example 5: Goal: Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} = \frac{A+B}{x+3}.$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2} \Longrightarrow 5x-2 = A(x+3)+B$$

$$\begin{aligned} x=-3: &\quad 5(-3)-2 = A((-3)+3)+B \\ \Longrightarrow &\quad -17 = 0+B \\ \Longrightarrow &\quad -17 = B \end{aligned}$$

$$\begin{aligned} 5x-2 &= A(x+3)-17 \\ &= Ax+3A-17 \\ \Longrightarrow \quad 5x &= Ax \\ \Longrightarrow \quad 5 &= A \end{aligned}$$

$$\int \frac{5x-2}{(x+3)^2}\,dx = \int \frac{5}{x+3} - \frac{17}{(x+3)^2}\,dx = \boxed{5\ln|x+3| + \frac{17}{x+3} + C}$$

Example 6: What if the denominator is an irreducible quadratic of the form x^2+px+q? That is, it can not be factored (does not have any real roots). In this case, suppose that $\left(x^2+px+q\right)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)}+\frac{B_1x+C_1}{(x^2+px+q)^2}+\frac{B_1x+C_1}{(x^2+px+q)^3}+\cdots+\frac{B_1x+C_1}{(x^2+px+q)^n}.$$

Compute $\displaystyle\int\frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

$$\frac{-2x+4}{(x^2+1)(x-1)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{(x-1)^2}=\frac{(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)+D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$\implies -2x+4=(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)+D(x^2+1)$$

There are four unknowns here, A, B, C and D. In this case we're going to want to minimise the amount of work we do here. In general it is going to be beneficial to solve for as many coefficients as we can by plugging in numbers, and then expand everything to compare coefficient after reducing the workload.

$$\begin{aligned}
x=1: &\quad -2(1)+4=(Ax+B)((1)-1)^2+C((1)^2+1)((1)-1)+D((1)^2+1)\\
\implies &\quad 2=0+0+2D\\
\implies &\quad 1=D
\end{aligned}$$

So we got one coefficient this way. That's better than nothing! Now if we use this new information and then rearrange a little we end up with less solving to do. This *does* require you however to be comfortable with algebra.

$$\begin{aligned}
\implies &\quad -2x+4=(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)+(x^2+1)\\
\implies &\quad -x^2-2x+3=(Ax+B)\,(x-1)^2+C(x^2+1)(x-1)\\
&\quad -(x^2+2x-3)=\\
&\quad -(x-1)(x+3)=
\end{aligned}$$

Now we have already seen what happens when $x=1$, so we can go right ahead and divide by the $(x-1)$ term that appears on both sides.

$$\begin{aligned}
\implies \quad -x-3&=(Ax+B)\,(x-1)+C(x^2+1)\\
&=Ax^2+Bx-Ax-B+Cx^2+C\\
&=(A+C)x^2+(B-A)x+C-B
\end{aligned}$$

Now we can go through and set up equations and solve by coefficients. When there are lots of coefficients it is a good idea of coming up with a way to book-keep your algebra - it can get *very* messy if you don't. Below is just one way you can do it.

$$\begin{pmatrix}(1) & 0 & = & A & & +C\\ (2) & -1 & = & -A & +B & \\ (3) & -3 & = & & -B & +C\end{pmatrix}\overset{(2)+(3)}{\implies}\begin{pmatrix}(1) & 0 & = & A & & +C\\ (2) & -4 & = & -A & & +C\\ (3) & -3 & = & & -B & +C\end{pmatrix}\overset{(1)+(2)}{\implies}\begin{pmatrix}(1) & -4 & = & & & 2C\\ (2) & -4 & = & -A & & +C\\ (3) & -3 & = & & -B & +C\end{pmatrix}$$

$$\begin{aligned}
\implies &\quad -4=2C &\implies &\quad -2=C\\
\implies &\quad -4=-A-2 &\implies &\quad 2=A\\
\implies &\quad -3=-B-2 &\implies &\quad 1=B,\ \text{So}\ldots
\end{aligned}$$

$$\begin{aligned}
\int\frac{-2x+4}{(x^2+1)(x-1)^2}\,dx&=\int\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\,dx\\
&=\int\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\,dx\\
&=\boxed{\ln(x^2+1)+\tan^{-1}(x)-2\ln|x-1|-\frac{1}{x-1}+C}
\end{aligned}$$

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper ($\deg f(x) < \deg g(x)$)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(X)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_1x + C_1}{(x^2+px+q)^2} + \frac{B_1x + C_1}{(x^2+px+q)^3} + \cdots + \frac{B_1x + C_1}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are
   (i) One fraction for each power of the irreducible factor that appears
   (ii) The degree of the numerator should be one less than the degree of the denominator

4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.