Section 8.5: Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that *can* be integrated using methods with which we are already familiar. That is, we cannot integrate

$$\frac{1}{x^2 - x}$$

as-is, but it is equivalent to

$$\frac{1}{x} - \frac{1}{x-1},$$

each term of which we can integrate.

Example 1: Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)}\, dx.$$

(a) $\displaystyle\int \frac{1}{x+1}\, dx =$

(b) $\displaystyle\frac{2}{x+1} - \frac{1}{x-3} =$

(c) $\displaystyle\int \frac{x-7}{(x+1)(x-3)}\, dx =$

Example 2: Compute $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx$.

(a) $\displaystyle\frac{7}{x-1} + \frac{3}{x-4} =$

(b) $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx =$

The previous two examples were nice since we were given a different expression of our integrand before hand. But what about when we don't? It is clear that the key step is decomposing our integrand into simple pieces, so how do we do it? The next example outlines the method.

Example 3: Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Example 4: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

Example 4 - An alternative approach: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

Example 5: Goal: Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} = \frac{A+B}{x+3}.$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

Example 6: What if the denominator is an irreducible quadratic of the form $x^2 + px + q$? That is, it can not be factored (does not have any real roots). In this case, suppose that $\left(x^2 + px + q\right)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{(x^2 + px + q)} + \frac{B_1 x + C_1}{\left(x^2 + px + q\right)^2} + \frac{B_1 x + C_1}{\left(x^2 + px + q\right)^3} + \cdots + \frac{B_1 x + C_1}{\left(x^2 + px + q\right)^n}.$$

Compute $\displaystyle\int \frac{-2x + 4}{x^2 + 1(x-1)^2}\, dx$.

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(X)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_1x + C_1}{(x^2+px+q)^2} + \frac{B_1x + C_1}{(x^2+px+q)^3} + \cdots + \frac{B_1x + C_1}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are

   (i) One fraction for each power of the irreducible factor that appears

   (ii) The degree of the numerator should be one less than the degree of the denominator

4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.