Section 8.4: Trigonometric Substitution

Motivation: If we want to find the area of a circle or ellipse, we have an integral of the form

$$\int \sqrt{a^2 - x^2}\, dx$$

where $a > 0$. Regular substitution will not work here, observe:

$$u = a^2 - x^2$$
$$du = -2x\, dx \longleftarrow \text{ extra factor of } x \ldots$$

Solution: Parametrise! We change x to a function of θ by letting $x = a\sin(\theta)$ so,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin(\theta))^2} = \sqrt{1^2 - a^2\sin^2(\theta)} = \sqrt{a^2\left(1 - \sin^2(\theta)\right)} = \sqrt{a^2\cos^2(\theta)} = a\left|\cos(\theta)\right|.$$

Generally, we use an injective (one-to-one) function (so it has an inverse) to simplify calculations. Above, we ensure $a\sin(\theta)$ is invertible by restricting the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Common Trig Substitutions: The following is a summary of when to use each trig substitution.

Integral contains:	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\left[0, \frac{\pi}{2}\right)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

If you are worried about remembering the identities, then don't! They can all be derived easily, assuming you know three basic ones (which by now you should):

$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1 \implies \cos^2(\theta) = 1 - \sin^2(\theta)$$

$$(\div \cos^2(\theta)) \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \implies \tan^2(\theta) = \sec^2(\theta) - 1$$

Example 1: Evaluate

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$$

$x = 3\sin(\theta)$

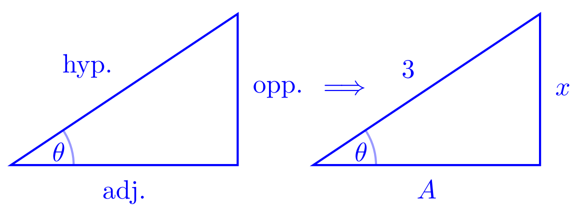
$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$dx = 3\cos(\theta)\ d\theta$

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{\sqrt{3^2 - 3^2\sin^2(\theta)}}{3^2\sin^2(\theta)} \cdot 3\cos(\theta)\ d\theta \\
&= \int \frac{\cancel{3}\sqrt{1-\sin^2(\theta)}}{\cancel{3^2}\sin^2(\theta)} \cdot \cancel{3}\cos(\theta)\ d\theta \\
&= \int \frac{\sqrt{\cos^2(\theta)}}{\sin^2(\theta)} \cdot \cos(\theta)\ d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \int \cot^2(\theta)\ d\theta \\
&= \int \csc^2(\theta) - 1\,d\theta \\
&= -\cot(\theta) - \theta + C \\
&= \boxed{-\frac{\sqrt{3^2-x^2}}{x} - \arcsin(\theta) + C}
\end{aligned}$$

How did we recover x?

$$x = 3\sin(\theta) \Longrightarrow \frac{x}{3} = \sin(\theta)$$



$A^2 + x^2 = 3^2$

$A^2 = 3^2 - x^2$

$A = \sqrt{3^2 - x^2}$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{3^2-x^2}}{x}$$

This is a common process in trig substitution. When you substitute back for your original variable, in this case x, you will always be able to find the correct substitutions by drawing out and labelling a right triangle correctly.

Example 2: Find

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

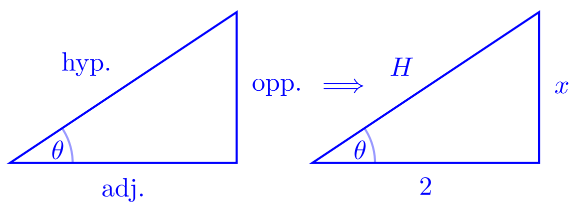
$$x = 2\tan(\theta)$$
$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$dx = 2\sec^2(\theta)\;d\theta$$

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{2\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{2^2\tan^2(\theta)+2^2}}\,d\theta \\
&= \int \frac{\cancel{2}\sec^2(\theta)}{2^2\tan^2(\theta)\,\cancel{2}\sqrt{\tan^2(\theta)+1}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec(\theta)}{2^2\tan^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{u^2}\,du \\
&= -\frac{1}{4}\frac{1}{u} + C \\
&= -\frac{1}{4\sin(\theta)} + C \\
&= -\frac{1}{4}\csc(\theta) + C \\
&= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}
\end{aligned}$$

$$u = \sin(\theta)$$
$$du = \cos(\theta)$$

How did we recover x?

$$x = 2\tan(\theta) \Longrightarrow \frac{x}{2} = \tan(\theta)$$



$$H^2 = x^2 + 2^2$$
$$H = \sqrt{x^2+4}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hyp.}}{\text{opp.}} = \frac{\sqrt{x^2+4}}{x}$$

Example 3: Evaluate

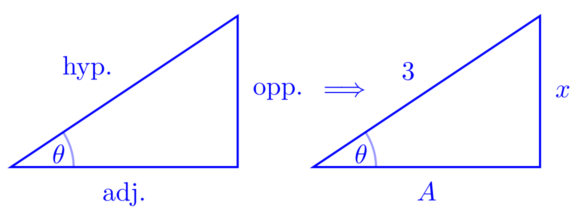
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

$$x = 3\sin(\theta)$$
$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$dx = 3\cos(\theta)\ d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{3^2\sin^2(\theta)}{\sqrt{3^2-3^2\sin^2(\theta)}}\cdot 3\cos(\theta)\ d\theta \\
&= \int \frac{3^2\sin^2(\theta)}{\cancel{3}\sqrt{1-\sin^2(\theta)}}\cdot \cancel{3}\cos(\theta)\ d\theta \\
&= \int \frac{3^2\sin^2(\theta)}{\sqrt{\cos^2(\theta)}}\cdot \cos(\theta)\ d\theta \\
&= 9\int \sin^2(\theta) \\
&= \frac{9}{2}\int 1-\cos(2\theta)\ d\theta \\
&= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{9}{2}\left(\theta - \sin(\theta)\cos(\theta)\right) + C \\
&= \boxed{\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C}
\end{aligned}$$

How did we recover x?

$$x = 3\sin(\theta) \Longrightarrow \frac{x}{3} = \sin(\theta)$$



$$3^2 = x^2 + A^2$$
$$A = \sqrt{9-x^2}$$

$$\cos(\theta) = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{9-x^2}}{x}$$