Section 8.3: Trigonometric Integrals - Worksheet

Goal: By using trig identities combined with u-substitution, we'd like to find antiderivatives of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

(for integer values of m and n). The goal of this worksheet[1] is for you to work together in groups of 2-3 to discover the techniques that work for these anti-derivatives.

Example 1 - Warm-up: Find

$$\int \cos^4(x)\sin(x)\,dx.$$

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$\begin{aligned}\int \cos^4(x)\sin(x)\,dx &= -\int u^4\,du \\ &= -\frac{u^5}{5} + C \\ &= \boxed{-\frac{\cos^5(x)}{5} + C}\end{aligned}$$

Example 2: Find

$$\int \sin^3(x)\,dx.$$

(Hint: Use the identity $\sin^2(x) + \cos^2(x) = 1$, then make a substitution.)

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \left(1-\cos^2(x)\right)\sin(x)\,dx \\ &= -\int \left(1-u^2\right)\,du \\ &= -u + \frac{u^3}{3} + C \\ &= \boxed{-\cos(x) + \frac{\cos^3(x)}{3} + C}\end{aligned}$$

[1]Worksheet adapted from BOALA, `math.colorado.edu/activecalc`

Example 3: Find

$$\int \sin^5(x)\cos^2(x)\,dx.$$

(Hint: Write $\sin^5(x)$ as $\left(\sin^2(x)\right)^2\sin(x)$.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,dx &= \int \left(\sin^2(x)\right)^2\cos^2(x)\sin(x)\,dx\\ &= \int \left(1-\cos^2(x)\right)^2\cos^2(x)\sin(x)\,dx\\ &= -\int \left(1-u^2\right)^2 u^2\,du\\ &= -\int \left(1-2u^2+u^4\right)\,du\\ &= -\int u^2-2u^4+u^6\,du\\ &= -\frac{u^3}{3}+\frac{2u^5}{5}-\frac{u^7}{7}+C\\ &= \boxed{-\frac{\cos^3(x)}{3}+\frac{2\cos^5(x)}{5}-\frac{\cos^7(x)}{7}+C}\end{aligned}$$

Example 4: Find

$$\int \sin^7(x)\cos^5(x)\,dx.$$

(The algebra here is long. Only set up the substitution - you do not need to fully evaluate.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}\int \sin^7(x)\cos^5(x)\,dx &= \int \left(\sin^2(x)\right)^3\cos^5(x)\sin(x)\,dx\\ &= \int \left(1-\cos^2(x)\right)^3\cos^5(x)\sin(x)\,dx\\ &= \boxed{-\int \left(1-u^2\right)^3 u^5\,du}\end{aligned}$$

Example 5: In general, how would you go about trying to find

$$\int \sin^m(x)\cos^n(x)\,dx,$$

where m is odd? (Hint: consider the previous three problems.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}\int \sin^m(x)\cos^n(x)\,dx &= \int \left(\sin^2(x)\right)^{(m-1)/2}\cos^n(x)\sin(x)\,dx\\ &= \int \left(1-\cos^2(x)\right)^{(m-1)/2}\cos^n(x)\sin(x)\,dx\\ &= \boxed{-\int \left(1-u^2\right)^{(m-1)/2} u^n\,du}\end{aligned}$$

Example 6: Note that the same kind of trick works when the power on $\cos(x)$ is odd. To check that you understand, what trig identity and what u-substitution would you use to integrate

$$\int \cos^3(x)\sin^2(x)\,dx?$$

$\sin^2(x)+\cos^2(x)=1$

$\boxed{\cos^2(x)=1-\sin^2(x)}$

$\boxed{\begin{array}{c} u=\sin(x) \\ du=\cos(x)\,dx \end{array}}$

$$\begin{aligned}\int \cos^3(x)\sin^2(x)\,dx &= \int \cos^2(x)\sin^2(x)\cos(x)\,dx \\ &= \int \left(1-\sin^2(x)\right)\sin^2(x)\cos(x)\,dx \\ &= \boxed{\int \left(1-u^2\right)u^2\,du}\end{aligned}$$

Example 7: Now what if the power on $\cos(x)$ and $\sin(x)$ are both even? Find

$$\int \sin^2(x)\,dx,$$

in each of the following two ways:

(a) Use the identity $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$.

$$\begin{aligned}\int \sin^2(x)\,dx &= \int \frac{1}{2}\left(1-\cos(2x)\right)\,dx \\ &= \frac{1}{2}\int 1-\cos(2x)\,dx \\ &= \boxed{\frac{1}{2}x-\frac{1}{4}\sin(2x)+C}\end{aligned}$$

(b) Integrate by parts, with $u=\sin(x)$ and $dv=\sin(x)\,dx$.

$u=\sin(x)$ $\quad dv=\sin(x)\,dx$

$du=\cos(x)\,dx$ $\quad v=-\cos(x)$

$$\begin{aligned}\int \sin^2(x)\,dx &= \int \sin(x)\sin(x)\,dx \\ &= -\sin(x)\cos(x)-\int -\cos(x)\cos(x)\,dx \\ &= -\sin(x)\cos(x)+\int \cos^2(x)\,dx \\ &= -\sin(x)\cos(x)+\int 1-\sin^2(x)\,dx \\ &= -\sin(x)\cos(x)+x-\int \sin^2(x)\,dx\end{aligned}$$

$$\Longrightarrow 2\int \sin^2(x)\,dx = -\sin(x)\cos(x)+x+C$$

$$\Longrightarrow \int \sin^2(x)\,dx = \boxed{\frac{x-\sin(x)\cos(x)}{2}+C}$$

(c) Show that your answers to parts (a) and (b) above are the same by giving a suitable trig identity.

$$\sin(x)\cos(x)=\frac{1}{2}2\sin(x)\cos(x)=\frac{1}{2}\sin(2x).$$

(d) How would you evaluate the integral

$$\int \sin^2(x)\cos^2(x)\,dx?$$

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}(1-\cos(2x)) \cdot \frac{1}{2}(1+\cos(2x))\,dx \\
&= \frac{1}{4}\int 1-\cos^2(x)\,dx \\
&= \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)\,dx \\
&= \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}(1+\cos(4x))\,dx \\
&= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos(4x)\,dx \\
&= \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}
\end{aligned}$$

Example 8: Evaluate the integral in problem (2) above, again, but this time by parts using $u = \sin^2(x)$ and $dv - \sin(x)\,dx$. (After this, you'll probably need to do a substitution.)

$u = \sin^2(x)$ $\quad dv = \sin(x)\,dx$

$du = 2\sin(x)\cos(x)\,dx$ $\quad v = -\cos(x)$

$w = \cos(x)$

$dw = -\sin(x)\,dx$

$$\begin{aligned}
\int \sin^3(x)\,dx &= \int \sin^2(x)\sin(x)\,dx \\
&= -\sin^2(x)\cos(x) - \int -\cos(x)\cdot 2\sin(x)\cos(x)\,dx \\
&= -\sin^2(x)\cos(x) + 2\int \cos^2(x)\sin(x)\,dx \\
&= -\sin^2(x)\cos(x) - 2\int w^2\,dw \\
&= -\sin^2(x)\cos(x) - \frac{2u^3}{3} + C \\
&= \boxed{-\sin^2(x)\cos(x) - \frac{2\cos^3(x)}{3} + C}
\end{aligned}$$

Example 9 - For fun: Can you show your answers to problem (2) and (8) above are the same? It's another great trigonometric identity.

$$-\sin^2(x)\cos(x) - \frac{2\cos^3(x)}{3} = -\left(1-\cos^2(x)\right)\cos(x) - \frac{2}{3}\cos^3(x) = -\cos(x) + \cos^3(x) - \frac{2}{3}\cos^3(x) = -\cos(x) + \frac{\cos^3(x)}{3}$$

Example 10 - Further investigations: (especially for mathematics, physics and engineering majors) We also would like to be able to solve integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

These two functions play well with each other, since the derivative of $\tan(x)$ is $\sec^2(x)$, the derivative of $\sec(x)$ is $\sec(x)|tan(x)$ and since there is a Pythagorean identity relating them. It sometimes works to use $u = \tan(x)$ and it sometimes works to use $u = \sec(x)$. Based on the values of m and n, which substitution should you use? Are there cases for which neither substitution works? (See page 472 of the text.)