# Section 8.2: Techniques of Integration

**A New Technique**: **Integration by parts** is a technique used to simplify integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when one of the functions ($f(x)$ or $g(x)$) can be differentiated repeatedly and the other function can be integrated repeatedly without difficulty. The following are two such integrals:

$$\int x\cos(x)\,dx \text{ and } \int x^2e^x\,dx.$$

Notice $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly (they are even eventually zero) and $g(x) = \cos(x)$ and $g(x) = e^x$ can be integrated repeatedly without difficulty.

**An Application of the Product Rule**: If $f(x)$ and $g(x)$ are differentiable functions of $x$, the product rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides and rearranging gives us the **Integration by Parts** formula!

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Longrightarrow \quad \int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

$$\Longrightarrow \quad \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

In differential form, let $u = f(x)$ and $v = g(x)$. Then,

**Integration by Parts Formula**:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Remember, all of the techniques that we talk about are supposed to make integrating easier! Even though this formula expresses one integral in terms of a second integral, the idea is that the second integral, $\int v\,du$, is easier to evaluate. The key to integration by parts is making the right choice for $u$ and $v$. Sometimes we may need to try multiple options before we can apply the formula.

**Example 1**: Find

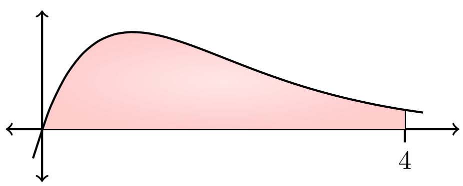
$$\int x\cos(x)\,dx.$$

We have to decide what to assign to $u$ and what to assign to $dv$. Our goal is to make the integral *easier*. One thing to bear in mind is that whichever term we let equal $u$ we need to differentiate - so if differentiating makes a part of the integrand simpler that's probably what we want! In this cases differentiating $\cos(x)$ gives $-\sin(x)$, which is no easier to deal with. But differentiating $x$ gives 1 which *is* simpler. So we have,

$$u = x \qquad dv = \cos(x)\,dx$$
$$du = dx \qquad v = \sin(x)$$

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx\\ &= \boxed{x\sin(x) + \cos(x) + C}\end{aligned}$$

**Example 3 - Integration by Parts for Definite Integrals**: Find the area of the region bounded by the curve $y = xe^{-x}$ and the $x$-axis from $x = 0$ to $x = 4$.




$$A = \int_0^4 xe^{-x}\,dx$$

$$u = x \qquad dv = e^{-x}\,dx$$
$$du = dx \qquad v = -e^{-x}$$

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big|_0^4 - \int_0^4 -e^{-x}\,dx\\ &= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx\\ &= \left(-4e^{-4} - 0\right) - e^{-x}\Big|_0^4\\ &= -4e^{-4} - \left(e^{-4} - 1\right)\\ &= -4e^{-4} - \left(e^{-4} - 1\right)\\ &= \boxed{-5e^{-4} + 1}\end{aligned}$$

**Example 3**: Evaluate

$$\int x^2 e^x \, dx.$$

Here we go through the same thought process. If $u = e^x$ then $du = e^x \, dx$, which doesn't make the problem any easier (though it doesn't make it any harder either). But in this case $dv = x^2$ would give $v = \frac{1}{3}x^3$ which arguably is *not* simpler that $x^2$. So,

$$u = x^2 \qquad dv = e^x \, dx$$
$$du = 2x \, dx \qquad v = e^x$$

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

It's at this point we see that we still cannot integrate the integral on the write easily. This is okay. **Sometimes we may have to apply the integration by parts formula more than once!**

$$u = x \qquad dv = e^x \, dx$$
$$du = dx \qquad v = e^x$$

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2\int x e^x \, dx \\ &= x^2 e^x - 2\left[x e^x - \int e^x \, dx\right] \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= \boxed{\left(x^2 - 2x + 2\right) e^x + C} \end{aligned}$$

The previous technique works for any integral of the form $\int x^n e^{mx} \, dx$, where $n$ is any *positive* integer and $m$ is any integer. What if $n$ was negative? Then this case we would set $u = e^x$.

**Example 4 - Tabular Method**: In Example 2 we have to apply the Integration by Parts Formula multiple times. There is a convenient way to "book-keep" our work. This is done by creating a table. Let's see how by examining Example 2 again.

Evaluate

$$\int x^2 e^x \, dx.$$

Let $f(x) = x^2$ and $g(x) = e^x$. Then,

| Differentiate $f(x)$ | | Integrate $g(x)$ |
|---|---|---|
| $x^2$ | $+$ | $e^x$ |
| $2x$ | $-$ | $e^x$ |
| $2$ | $+$ | $e^x$ |
| $0$ | | $e^x$ |

Then the integral is,

$$\int x^2 e^x \, dx = +x^2 \cdot e^x - 2x \cdot e^x + 2 \cdot e^x + C = \boxed{\left(x^2 - 2x + 2\right) e^x + C}$$

We *have* actually used the integration by parts formula, but we have just made our lives easier by condensing the work into a neat table. This method is extremely useful when Integration by Parts needs to be used over and over again.

**Example 5 - Recurring Integrals**: Find the integral

$$\int e^x \sin(x)\,dx.$$

We need to apply Integration by Parts twice before we see something:

(1)

$u = e^x \qquad dv = \sin(x)$

$du = e^x\,dx \qquad v = -\cos(x)$

(2)

$u = e^x \qquad dv = \cos(x)$

$du = e^x\,dx \qquad v = \sin(x)$

$$\begin{aligned}\int e^x \sin(x)\,dx &= -e^x\cos(x) + \int e^x\cos(x)\,dx \\ &= -e^x\cos(x) + \left(e^x\sin(x) - \int e^x\sin(x)\,dx\right) \\ &= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx\end{aligned}$$

Notice that now the integral we are interested in, $\int e^x\sin(x)\,dx$, appears on both the left and right hand side of the equation. So, if we add this integral to both sides we get

$$\implies \quad 2\int e^x\sin(x)\,dx = e^x\left(-\cos(x)+\sin(x)\right)$$

$$\implies \quad \int e^x\sin(x)\,dx = \boxed{\frac{e^x\left(\sin(x)-\cos(x)\right)}{2}}$$

This "trick" comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like $e^x$ and trig functions indefinitely and never reach 0. Polynomials on the other hand will eventually "terminate" and their $n^{\text{th}}$ derivative (where $n$ is the degree of the polynomial) is identically 0.

**Example 6 - Challenge**: Find the integral

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos(nx)\, dx,$$

where $n$ is a positive integer.

Let $f(x) = x^2$ and $g(x) = \cos(nx)$. Then,

| Differentiate $f(x)$ | | Integrate $g(x)$ |
|---|---|---|
| $x^3$ | $+$ | $\cos(nx)$ |
| $3x^2$ | $-$ | $\frac{1}{n}\sin(nx)$ |
| $6x$ | $+$ | $-\frac{1}{n^2}\cos(nx)$ |
| $6$ | $-$ | $-\frac{1}{n^3}\sin(nx)$ |
| $0$ | | $\frac{1}{n^4}\cos(nx)$ |

Then the integral is,

$$\begin{aligned}
\frac{1}{\pi}\int x^3\cos(nx)\,dx &= \frac{1}{\pi}\left[+x^3\cdot\frac{1}{n}\sin(nx) - 3x^2\cdot\left(-\frac{1}{n^2}\right)\cos(nx) + 6x\cdot\left(-\frac{1}{n^3}\right)\sin(nx) - 6\cdot\frac{1}{n^4}\cos(nx)\right]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{x^3}{n}\sin(nx) + \frac{3x^2}{n^2}\cos(nx) - \frac{6x}{n^3}\sin(nx) - \frac{6}{n^4}\cos(nx)\right]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\left(0 + \frac{3\pi^2}{n^2}\cos(n\pi) - 0 + \frac{6}{n^4}\right) - \left(0 + 0 - 0 - \frac{6}{n^4}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{3\pi^2(-1)^n}{n^2} - \frac{6(-1)^n}{n^4} + \frac{6}{n^4}\right] \\
&= \boxed{\frac{3}{\pi}\,\frac{\pi^2 n^2(-1)^n - 2(-1)^n + 2}{n^4}}
\end{aligned}$$