Section 8.1: Using Basic Integration Formulas

A Review: The basic integration formulas summarise the forms of indefinite integrals for may of the functions we have studied so far, and the substitution method helps us use the table below to evaluate more complicated functions involving these basic ones. So far, we have seen how to apply the formulas directly and how to make certain u-substitutions. Sometimes we can rewrite an integral to match it to a standard form. More often however, we will need more advanced techniques for solving integrals. First, let's look at some examples of our known methods.

Basic integration formulas			
1. $\int k\,dx = kx + C$	(any number k)	12. $\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$(n \neq -1)$	13. $\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$	
3. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$		14. $\int \sec(x)\,dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$	
4. $\int e^x\,dx = e^x + C$		15. $\int \csc(x)\,dx = -\ln\lvert\csc(x) + \cot(x)\rvert + C$	
5. $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$	$(a > 0,\ a \neq 1)$	16. $\int \sinh(x)\,dx = \cosh(x) + C$	
6. $\int \sin(x)\,dx = -\cos(x) + C$		17. $\int \cosh(x)\,dx = \sinh(x) + C$	
7. $\int \cos(x)\,dx = \sin(x) + C$		18. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
8. $\int \sec^2(x)\,dx = \tan(x) + C$		19. $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
9. $\int \csc^2(x)\,dx = -\cot(x) + C$		20. $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$	$(a > 0)$
10. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$		21. $\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
11. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$		22. $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$	$(x > a > 0)$

Example 1 - Substitution: Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

$u = x^2 - 3x + 1$

$du = 2x - 3\,dx$

$u = (3)^2 - 3(3) + 1 = 1$

$u = (5)^2 - 3(5) + 1 = 11$

$$\begin{aligned}\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx &= \int_1^{11} \frac{1}{\sqrt{u}}\,du \\ &= \int_1^{11} u^{-1/2}\,du \\ &= 2u^{1/2}\Big|_1^{11} \\ &= 2\sqrt{11} - 2\sqrt{1} \\ &= \boxed{2\left(\sqrt{11}-1\right)}\end{aligned}$$

Example 2 - Complete the Square: Find

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx.$$

$$\begin{aligned}8x - x^2 &= -(x^2 - 8x) \\ &= -((x-4)^2 - 4^2) \\ &= 4^2 - (x-4)^2\end{aligned}$$

$u = x - 4$

$du = dx$

$$\begin{aligned}\int \frac{1}{\sqrt{8x-x^2}}\,dx &= \int \frac{1}{\sqrt{4^2-(x-4)^2}}\,dx \\ &= \int \frac{1}{\sqrt{4^2-(u)^2}}\,du \\ &= \sin^{-1}\left(\frac{u}{4}\right) + C \\ &= \boxed{\sin^{-1}\left(\frac{x-4}{4}\right) + C}\end{aligned}$$

Example 3 - Trig Identities: Calculate

$$\int \cos(x)\sin(2x) + \sin(x)\cos(2x)\,dx.$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$\begin{aligned}\int \cos(x)\sin(2x) + \sin(x)\cos(2x)\,dx &= \int \sin(x+2x)\,dx \\ &= \int \sin(3x)\,dx \\ &= \int \frac{1}{3}\sin(u)\,du \\ &= -\frac{1}{3}\cos(u) + C \\ &= \boxed{-\frac{1}{3}\cos(3x) + C}\end{aligned}$$

Example 4 - Trig Identities: Find

$$\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\,dx.$$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\,dx &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)}\,dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\sin(x)}{1-\sin^2(x)}\,dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(x)} + \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}\,dx \\
&= \int_0^{\frac{\pi}{4}} \sec^2(x) + \sec(x)\tan(x)\,dx \\
&= \tan(x) + \sec(x)\Big|_0^{\frac{\pi}{4}} \\
&= \tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) - (\tan(0) + \sec(0)) \\
&= 1 + \sqrt{2} - (0+1) \\
&= \boxed{\sqrt{2}}
\end{aligned}$$

Example 5 - Clever Substitution Evaluate

$$\int \frac{1}{(1+\sqrt{x})^3}\,dx.$$

$$\begin{aligned}
u &= 1+\sqrt{x} \\
du &= \frac{1}{2\sqrt{x}}\,dx \\
2\sqrt{x}\,du &= dx \\
2(u-1)\,du &= dx
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{(1+\sqrt{x})^3}\,dx &= \int \frac{2(u-1)}{u^3}\,du \\
&= \int \frac{2}{u^2} - \frac{2}{u^3}\,du \\
&= \int 2u^{-2} - 2u^{-3}\,du \\
&= -2u^{-1} + u^{-2} + C \\
&= -\frac{2}{u} + \frac{1}{u^2} + C \\
&= \boxed{-\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C}
\end{aligned}$$

Example 6 - Properties of Trig Integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\cos(x)\,dx.$$

$$f(x) = x^3 \implies f(-x) = (-x)^3 = -x^3 = -f(x) \implies x^3 \text{ is an odd function}$$

$$g(x) = \cos(x) \implies f(-x) = \cos(-x) = \cos(x) = f(x) \implies x^3 \text{ is an even function}$$

Putting these two facts together we see that $x^3\cos(x)$ is an odd function and is symmetric over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus (by Theorem 8, Section 5.6)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\cos(x)\,dx = \boxed{0}$$