Section 8.1: Using Basic Integration Formulas

A Review: The basic integration formulas summarise the forms of indefinite integrals for may of the functions we have studied so far, and the substitution method helps us use the table below to evaluate more complicated functions involving these basic ones. So far, we have seen how to apply the formulas directly and how to make certain u-substitutions. Sometimes we can rewrite an integral to match it to a standard form. More often however, we will need more advanced techniques for solving integrals. First, let's look at some examples of our known methods.

Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)	12. $\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$	13. $\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$
3. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	14. $\int \sec(x)\,dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$
4. $\int e^x\,dx = e^x + C$	15. $\int \csc(x)\,dx = -\ln\lvert\csc(x) + \cot(x)\rvert + C$
5. $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$ $(a > 0,\, a \neq 1)$	16. $\int \sinh(x)\,dx = \cosh(x) + C$
6. $\int \sin(x)\,dx = -\cos(x) + C$	17. $\int \cosh(x)\,dx = \sinh(x) + C$
7. $\int \cos(x)\,dx = \sin(x) + C$	18. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
8. $\int \sec^2(x)\,dx = \tan(x) + C$	19. $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
9. $\int \csc^2(x)\,dx = -\cot(x) + C$	20. $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$ $(a > 0)$
10. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$	21. $\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
11. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$	22. $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$

Example 1 - Substitution: Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

Example 2 - Complete the Square: Find

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx.$$

Example 3 - Trig Identities: Calculate

$$\int \cos(x)\sin(2x)+\sin(x)\cos(2x)\,dx.$$

Example 4 - Trig Identities: Find

$$\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\,dx.$$

Example 5 - Clever Substitution Evaluate

$$\int \frac{1}{(1+\sqrt{x})^3}\,dx.$$

Example 6 - Properties of Trig Integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\cos(x)\,dx.$$

Section 8.2: Techniques of Integration

A New Technique: Integration is a technique used to simplify integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when one of the functions ($f(x)$ or $g(x)$) can be differentiated repeatedly and the other function can be integrated repeatedly without difficulty. The following are two such integrals:

$$\int x\cos(x)\,dx \text{ and } \int x^2e^x\,dx.$$

An Application of the Product Rule: If $f(x)$ and $g(x)$ are differentiable functions of x, the product rule says that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides and rearranging gives us the **Integration by Parts** formula!

In differential form, let $u = f(x)$ and $v = g(x)$. Then,

Integration by Parts Formula:

Remember, all of the techniques that we talk about are supposed to make integrating easier! Even though this formula expresses one integral in terms of a second integral, the idea is that the second integral, $\int v\,du$, is easier to evaluate. The key to integration by parts is making the right choice for u and v. Sometimes we may need to try multiple options before we can apply the formula.

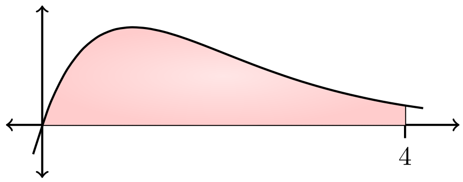
Example 1: Find

$$\int x \cos(x)\, dx.$$

Example 2: Evaluate

$$\int x^2 e^x\, dx.$$

Example 3 - Integration by Parts for Definite Integrals: Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



Example 4 - Tabular Method: In Example 2 we have to apply the Integration by Parts Formula multiple times. There is a convenient way to "book-keep" our work. This is done by creating a table. Let's see how by examining Example 2 again.

Evaluate

$$\int x^2 e^x \, dx.$$

Example 5: Find the integral

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos(nx)\ dx,$$

where n is a positive integer.

Example 6 - Recurring Integrals: Find the integral

$$\int e^x \sin(x)\, dx.$$

This "trick" comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like e^x and trig functions indefinitely and never reach 0. Polynomials on the other hand will eventually "terminate" and their n^{th} derivative (where n is the degree of the polynomial) is identically 0.

Section 8.3: Trigonometric Integrals - Worksheet

Goal: By using trig identities combined with u-substitution, we'd like to find antiderivatives of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

(for integer values of m and n). The goal of this worksheet[1] is for you to work together in groups of 2-3 to discover the techniques that work for these anti-derivatives.

Example 1 - Warm-up: Find

$$\int \cos^4(x)\sin(x)\,dx.$$

Example 2: Find

$$\int \sin^3(x)\,dx.$$

(Hint: Use the identity $\sin^2(x) + \cos^2(x) = 1$, then make a substitution.)

[1]Worksheet adapted from BOALA, `math.colorado.edu/activecalc`

Example 3: Find

$$\int \sin^5(x)\cos^2(x)\,dx.$$

(Hint: Write $\sin^5(x)$ as $\left(\sin^2(x)\right)^2\sin(x)$.)

Example 4: Find

$$\int \sin^7(x)\cos^5(x)\,dx.$$

(The algebra here is long. Only set up the substitution - you do not need to fully evaluate.)

Example 5: In general, how would you go about trying to find

$$\int \sin^m(x)\cos^n(x)\,dx,$$

where m is odd? (Hint: consider the previous three problems.)

Example 6: Note that the same kind of trick works when the power on $\cos(x)$ is odd. To check that you understand, what trig identity and what u-substitution would you use to integrate

$$\int \cos^3(x)\sin^2(x)\,dx?$$

Example 7: Now what if the power on $\cos(x)$ and $\sin(x)$ are both even? Find

$$\int \sin^2(x)\,dx,$$

in each of the following two ways:

(a) Use the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

(b) Integrate by parts, with $u = \sin(x)$ and $dv = \sin(x)\,dx$.

(c) Show that your answers to parts (a) and (b) above are the same by giving a suitable trig identity.

(d) How would you evaluate the integral

$$\int \sin^2(x)\cos^2(x)\,dx?$$

Example 8: Evaluate the integral in problem (2) above, again, but this time by parts using $u = \sin^2(x)$ and $dv - \sin(x)\,dx$. (After this, you'll probably need to do a substitution.)

Example 9 - For fun: Can you show your answers to problem (2) and (8) above are the same? It's another great trigonometric identity.

Example 10 - Further investigations: (especially for mathematics, physics and engineering majors) We also would like to be able to solve integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

These two functions play well with each other, since the derivative of $\tan(x)$ is $\sec^2(x)$, the derivative of $\sec(x)$ is $\sec(x)|tan(x)$ and since there is a Pythagorean identity relating them. It sometimes works to use $u = \tan(x)$ and it sometimes works to use $u = \sec(x)$. Based on the values of m and n, which substitution should you use? Are there cases for which neither substitution works? (See page 472 of the text.)

Section 8.4: Trigonometric Substitution

Motivation: If we want to find the area of a circle or ellipse, we have an integral of the form

$$\int \sqrt{a^2 - x^2}\, dx$$

where $a > 0$. Regular substitution will not work here, observe:

$$\begin{aligned} u &= a^2 - x^2 \\ du &= -2x\, dx \longleftarrow \text{ extra factor of } x \ldots \end{aligned}$$

Solution: Parametrise! We change x to a function of θ by letting $x = a\sin(\theta)$ so,

Generally, we use an injective (one-to-one) function (so it has an inverse) to simplify calculations. Above, we ensure $a\sin(\theta)$ is invertible by restricting the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Common Trig Substitutions: The following is a summary of when to use each trig substitution.

Integral contains:	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\left[0, \frac{\pi}{2}\right)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

If you are worried about remembering the identities, then don't! They can all be derived easily, assuming you know three basic ones (which by now you should):

$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Example 1: Evaluate

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$$

Example 2: Find

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

Example 3: Evaluate

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

Section 8.5: Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that *can* be integrated using methods with which we are already familiar. That is, we cannot integrate

$$\frac{1}{x^2 - x}$$

as-is, but it is equivalent to

$$\frac{1}{x} - \frac{1}{x-1},$$

each term of which we can integrate.

Example 1: Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)}\, dx.$$

(a) $\displaystyle\int \frac{1}{x+1}\, dx =$

(b) $\displaystyle\frac{2}{x+1} - \frac{1}{x-3} =$

(c) $\displaystyle\int \frac{x-7}{(x+1)(x-3)}\, dx =$

Example 2: Compute $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx$.

(a) $\displaystyle\frac{7}{x-1} + \frac{3}{x-4} =$

(b) $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx =$

The previous two examples were nice since we were given a different expression of our integrand before hand. But what about when we don't? It is clear that the key step is decomposing our integrand into simple pieces, so how do we do it? The next example outlines the method.

Example 3: Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Example 4: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

Example 4 - An alternative approach: Find

$$\int \frac{x+15}{(3x-4)\,(x+1)}\,dx.$$

Example 5: Goal: Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} = \frac{A+B}{x+3}.$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

Example 6: What if the denominator is an irreducible quadratic of the form $x^2 + px + q$? That is, it can not be factored (does not have any real roots). In this case, suppose that $\left(x^2 + px + q\right)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{(x^2 + px + q)} + \frac{B_1 x + C_1}{(x^2 + px + q)^2} + \frac{B_1 x + C_1}{(x^2 + px + q)^3} + \cdots + \frac{B_1 x + C_1}{(x^2 + px + q)^n}.$$

Compute $\displaystyle\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx.$

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper

1. Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(X)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_1x+C_1}{(x^2+px+q)^2} + \frac{B_1x+C_1}{(x^2+px+q)^3} + \cdots + \frac{B_1x+C_1}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are

   (i) One fraction for each power of the irreducible factor that appears

   (ii) The degree of the numerator should be one less than the degree of the denominator

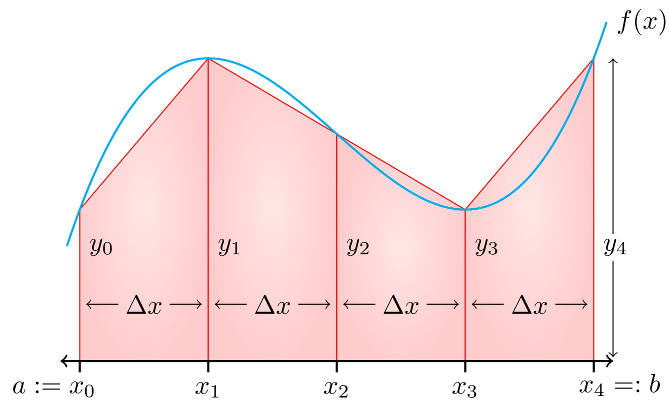
4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

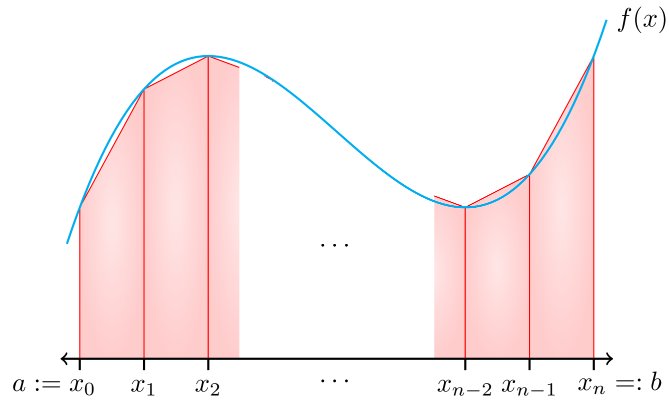
5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.

Section 8.7: Numerical Integration

What to do when there's no nice antiderivative? The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln(x)$ and $\sqrt{1+x^4}$ have no elementary formulas/ When we cannot find a workable antiderivative for a function $f(x)$ that we have to integrate, we can partition the interval of integration, replace $f(x)$ by a closely fitting polynomial on each subinterval, integrate the poynomials and add the results to *approximate* the definite integral of $f(x)$. This is an example of numerical integration. There are many methods of numerical integration but we will study only two: the *Trapezium Rule* and *Simpson's Rule*.

Trapezoidal Approximations: As the name implies, the Trapezium Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapeziums instead of rectangles - which, if you recall, we studied when we looked at Riemann integration in Calculus I.





The Trapezium Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f\left(x_i\right)\right), \end{aligned}$$

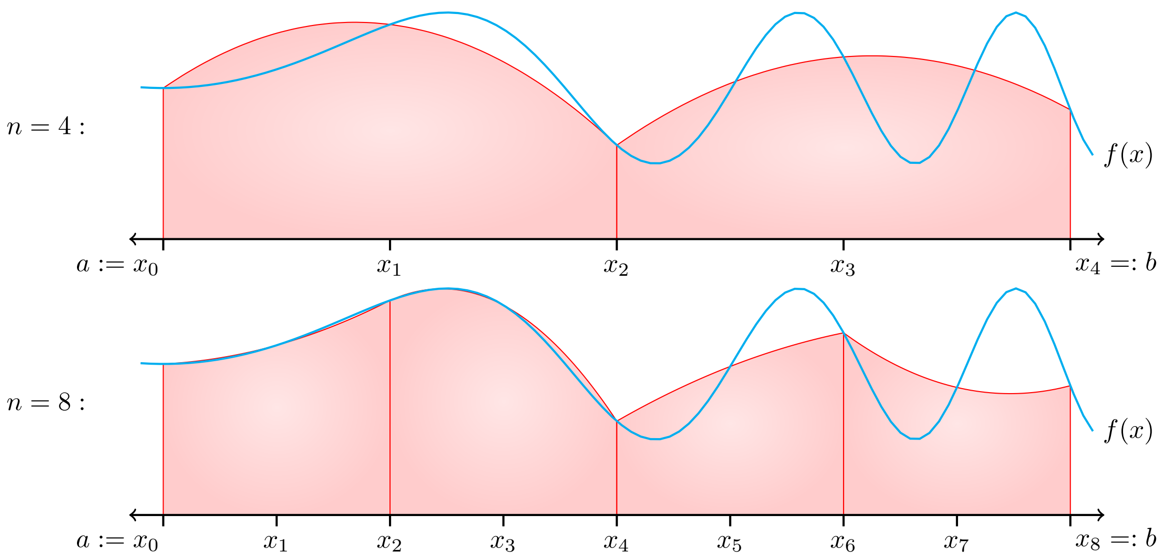
where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$.

Example 1: Use the Trapezium Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Parabolic Approximations: Instead of using the straight-line segments that produced the trapeziums, we can use parabolas to approximate the definite integral of a continuous function. We partition the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$ but this time we require n to be an even number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola. A typical parabola passed through three consecutive points: (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) on the curve.



Simpson's Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{3}\left(f\left(x_0\right) + f\left(x_n\right) + 2\left(\sum_{i=1}^{\frac{n-1}{2}} f\left(x_{2i-1}\right) + 2f\left(x_{2i}\right)\right)\right), \end{aligned}$$

where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$ with n an *even* number.

Example 2: Use the Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$. Compare the estimate with the exact value.

Error Estimates in the Trapezium and Simpson's Rules If $f''(x)$ is continuous and M is any upper bound for the values of $|f''(x)|$ on $[a,b]$, then the error E_T in the Trapezium Rule for approximating the definite integral of $f(x)$ over the interval $[a,b]$ using n trapeziums satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

If $f^{(4)}(x)$ is continuous and M is any upper bound for the values of $|f^{(4)}(x)|$ on $[a,b]$, then the error E_S in Simpson's Rule for approximating the definite integral of $f(x)$ over the interval $[a,b]$ using $\frac{n}{2}$ parabolas satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

Example 3: Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$. What value of n should we pick so that the error is within 0.001 of the true value?

Section 8.8: Improper Integrals

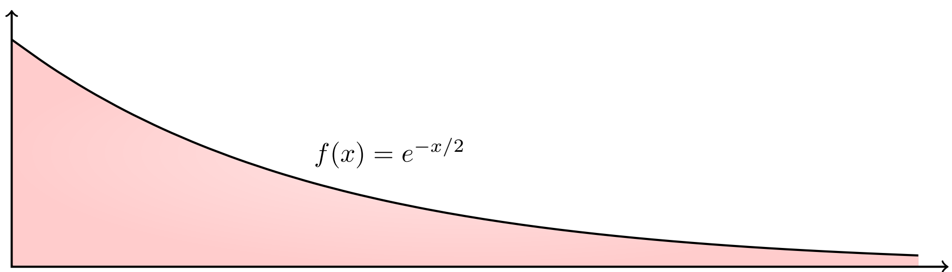
Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite

2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty} \int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

   where c is any real number.

In each case, if the limit is finite we sat that the improper integral ______________________ and that the limit is the

__________________ of the improper integral. If the limit fails to exist, the improper integral _____________________

Any of the integrals in the above definition can be interpreted as an area if $f(x) \geq 0$ on the interval of integration. If $f(x) \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

A Special Example: For what values of p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^a f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral ________________________ and that the limit is the

____________________ of the improper integral. If the limit fails to exist, the improper integral ______________________

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.