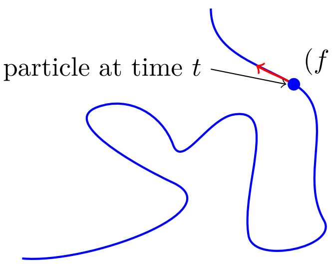
Section 11.1: Parametrisations of Plane Curves

Parametric Equations: Below we have the path of a moving particle on the xy-plane. We can sometimes describe such a path by a pair of equations, $x = f(t)$ and $y = g(t)$, where $f(t)$ and $g(t)$ are continuous functions. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = (f(t), g(t))$ at any time t.



Definitions: If x and y are given as functions

$$x = f(t) \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve.

The equations are parametric equations for the curve.

The variable t is the parameter for the curve and its domain I is the parameter interval.

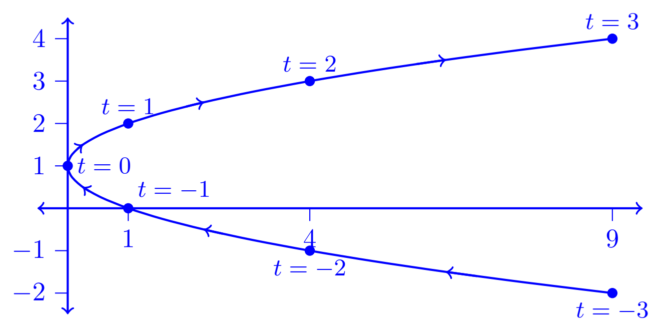
If I is a closed interval, $a \le t \le b$, the initial point of the curve is the point $(f(a), g(a))$ and the terminal point of the curve is $(f(b), g(b))$.

Example 1: Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

The (x, y) coordinates are determined by values for t, $(t^2, t + 1)$.

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



Here the arrows indicate the direction of travel.

Example 2: Identify geometrically the curve in Example 1 by eliminating the parameter t and obtaining an algebraic equation in x and y.

Since both x and y are defined in terms of t, we can use substitution to eliminate the parameter:

Option 1:

$$y = t + 1 \qquad x = t^2$$
$$\implies y - 1 = t \qquad x = (y-1)^2$$
$$\implies \boxed{x = y^2 - 2y + 1}$$

Option 2:

$$x = t^2 \qquad y = t + 1$$
$$\implies \pm\sqrt{x} = t \qquad y = \pm\sqrt{x} + 1$$
$$\implies \boxed{y = \sqrt{x} + 1, \quad y = -\sqrt{x} + 1}$$

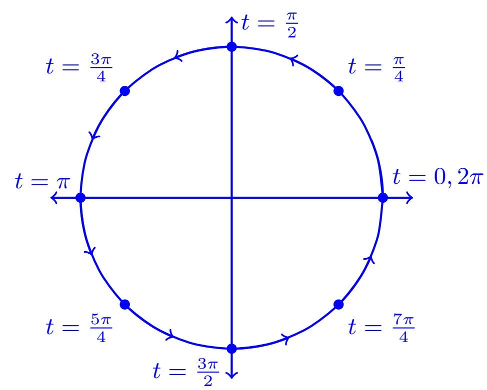
Example 3: Graph the parametric curves

(a) $x = \cos(t), \qquad y = \sin(t), \qquad 0 \le t \le 2\pi,$

(b) $x = a\cos(t), \quad y = a\sin(t), \quad 0 \le t \le 2\pi, \quad a \in \mathbb{R}.$

(a)

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
π	-1	0
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
2π	1	0



Here the arrows indicate the direction of travel.

We see then that these parametric equations correspond to travelling around the unit circle anticlockwise. Algebraically we can verify this to see that

$$\cos^2(t) + \sin^2(t) = x^2 + y^2 = 1$$

which is precisely the equation for a circle of radius 1, centred at the origin.

(b) It should come at no surprise that these parametric equations correspond to travelling around the circle of radius a, centred at the origin, anticlockwise.

Example 4: The position $P(x,y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

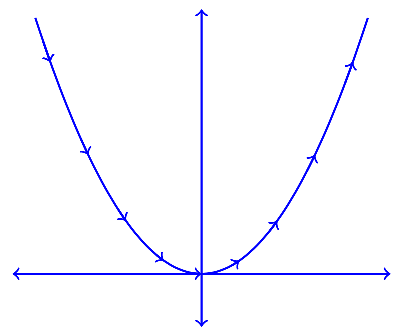
$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

We can either find a table of values and plot or we can find a Cartesian equation. The latter is more straight forward and we see that $x = \sqrt{y}$ for $y \geq 0$ (or $y = x^2$ for $x \geq 0$). So the curve is the part of $y = x^2$ lying in the first quadrant of the xy-plane.

Example 5 - Natural Parametrisation: A parametrisation of the function $f(x) = x^2$ is given by

Let $x = t$. Then $y = x^2 = t^2$ and so the *natural parametrisation* of the curve $y = x^2$ is (t, t^2) where $-\infty < t < \infty$.



Example 6: Find a parametrisation for the line through the point (a,b) having slope m.

A Cartesian equation of the line through (a,b) with slope m is

$$y - b = m(x - a).$$

Let $t = x - a$. Then $y - b = mt$ so $y = mt + b$. Therefore a parametrisation is

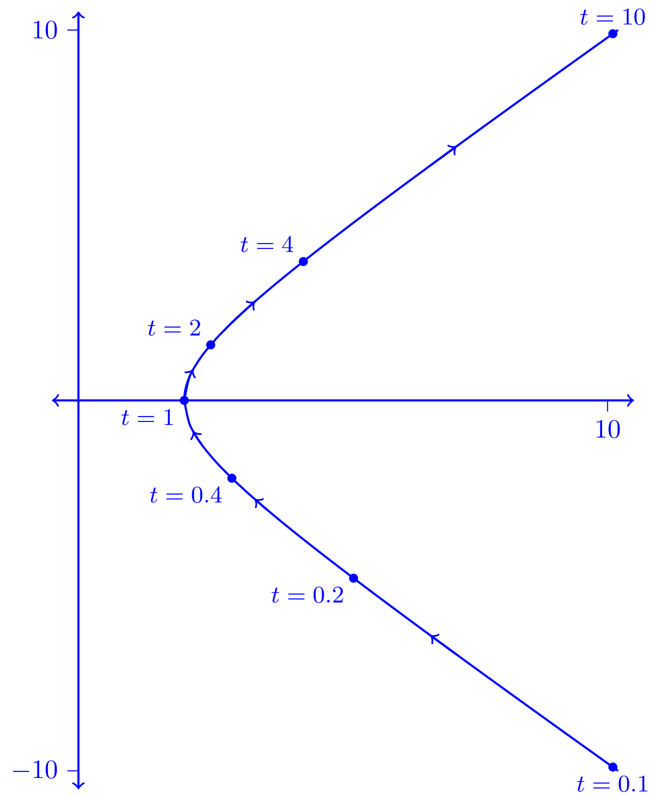
$$(x,y) = (t + a, mt + b), \quad -\infty < t < \infty.$$

It is important that the usage of the phrase "a parametrisation" is precise here since parametrisations are not unique. Here we could also use the *natural parametrisation* to obtain $(x,y) = (t, mt - (ma - b))$, $-\infty < t < \infty$.

Example 7: Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

t	x	y
0.1	10.1	-9.9
0.2	5.2	-4.8
0.4	2.9	2.1
1	2	0
2	2.5	1.5
4	4.25	3.75
10	10.1	9.9



(1) $x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t$

(2) $x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}$

(3) $x^2 - y^2 = (x+y)(x-y) = (2t)\left(\frac{2}{t}\right) = 4$

The Cartesian equation $x^2 - y^2 = 4$ is the standard form for the equation of a hyperbola.

Section 11.2: Calculus with Parametric Equations

Tangents and Areas: A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If all three derivatives exist and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Further we also have

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt}.$$

Example 1: Find the tangent to the curve

$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$.

First we need to calculate the value of t at the point $(\sqrt{2}, 1)$. Since $\tan(x)$ is a one-to-one function on the parameter interval we see that

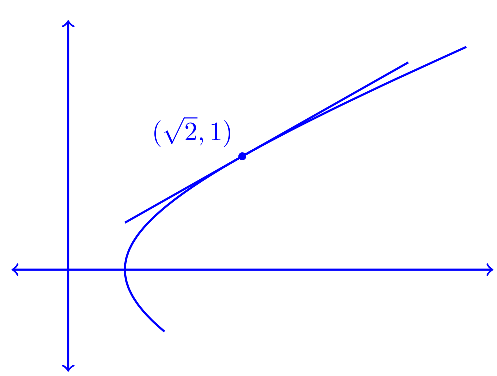
$$t = \tan^{-1}(1) = \frac{\pi}{4}$$

Using this we calculate the slope of the tangent line.

$$m = \frac{dy}{dx}\Big|_{t=\frac{\pi}{4}} = \frac{dy/dt}{dx/dt}\Big|_{t=\frac{\pi}{4}} = \frac{\sec^2(t)}{\sec(t)\tan(t)}\Big|_{t=\frac{\pi}{4}} = \frac{\sec(t)}{\tan(t)}\Big|_{t=\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Thus the equation of the tangent line at the point $(\sqrt{2}, 1)$ is

$$\boxed{y = \sqrt{2}(x - \sqrt{2}) + 1}$$



Example 2: Find $\frac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

$$\frac{dx}{dt} = 1 - 2t \qquad \frac{dy}{dt} = 1 - 3t^2$$

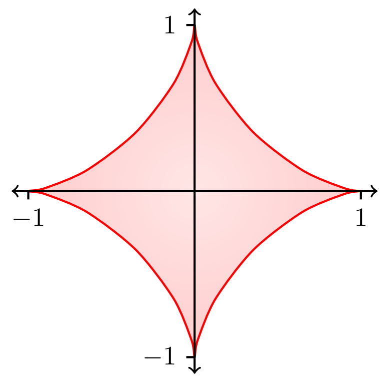
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dt}\right) &= \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)\\ &= \frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}\\ &= \frac{-6t+12t^2+2-6t^2}{(1-2t)^2}\\ &= \frac{2-6t+6t^2}{(1-2t)^2}\end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt} = \boxed{\frac{2-6t+6t^2}{(1-2t)^3}}$$

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$



The shape we are dealing with is symmetric, so the area we are interested in is four times the area beneath the curve in the first quadrant, corresponding to $0 \le t \le \frac{\pi}{2}$. We will apply the Fundamental Theorem of Calculus using substitution to express the curve y ad the differential dx in terms of t.

$$x = \cos^3(t)$$
$$dx = -3\cos^2(t)\sin(t)\,dt$$

$$\begin{aligned}A = 4\int_0^1 y(x)\,dx &= 4\int_{\frac{\pi}{2}}^0 \sin^3(t)\left(-3\cos^2(t)\sin(t)\right)\,dt\\ &= 12\int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)\,dt\\ &= 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2t)}{2}\right)^2\left(\frac{1+\cos(2t)}{2}\right)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2t))^2(1+\cos(2t))\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2t))\left(1-\cos^2(2t)\right)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin^2(2t)-\cos(2t)\sin^2(2t)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4t)}{2}\,dt - \frac{3}{2}\int_{t=0}^{t=\frac{\pi}{2}} \frac{u^2}{2}\,du\\ &= \frac{3}{4}\left[t-\frac{1}{4}\sin(4t)\right]_0^{\frac{\pi}{2}} - \frac{3}{4}\left[\frac{u^3}{3}\right]_{t=0}^{t=\frac{\pi}{2}}\\ &= \frac{3}{4}\left[t-\frac{1}{4}\sin(4t)-\frac{1}{3}\sin^3(2t)\right]_0^{\frac{\pi}{2}}\\ &= \boxed{\frac{3\pi}{8}}\end{aligned}$$

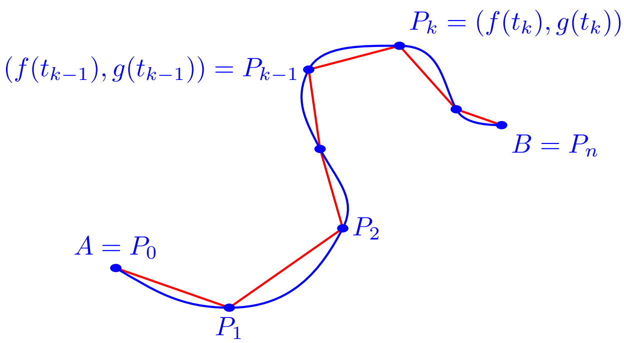
$$u = \sin(2t)$$
$$du = 2\cos(2t)\,dt$$

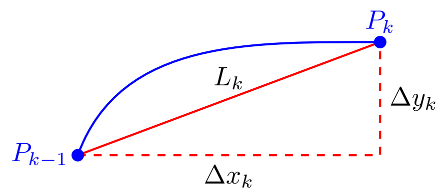
Length of a Parametrically Defined Curve: Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

We assume the functions $f(t)$ and $g(t)$ are continuously differentiable on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a smooth curve.





The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment shown on the right, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \text{ and } g'(t_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes

$$L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

In an surprising turn of events, we obtain the exact value of L by taking a limit of this sum, resulting in a definite integral. To summarise:

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 4: Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \le t \le 2\pi.$$

$$\frac{dx}{dt} = -r\sin(t)$$

$$\frac{dy}{dt} = r\cos(t)$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2\left(\sin^2(t) + \cos^2(t)\right)}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2}\, dt \\ &= \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = \boxed{2\pi r} \end{aligned}$$

Example 5: Find the length of the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$

As in Example 3, the perimeter of the astroid is 4 times the length of the curve in the first quadrant.

$$\frac{dx}{dt} = 3\cos^2(t)\sin(t)$$

$$\frac{dy}{dt} = -3\cos^2(t)\sin(t)$$

$$u = \sin(t)$$

$$du = \cos(t)\, dt$$

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2(t)\sin^2(t)\left(\cos^2(t) + \sin^2(t)\right)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} 3\cos(t)\sin(t)\, dt \\ &= 12\int_{t=0}^{t=\frac{\pi}{2}} u\, du \\ &= 12\left[\frac{u^2}{2}\right]_{t=0}^{t=\frac{\pi}{2}} \\ &= 12\left[\frac{\sin^2(t)}{2}\right]_0^{\frac{\pi}{2}} \\ &= 12\left[\frac{1}{2} - 0\right] \\ &= \boxed{6} \end{aligned}$$

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \leq t \leq 2\pi.$$

Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.

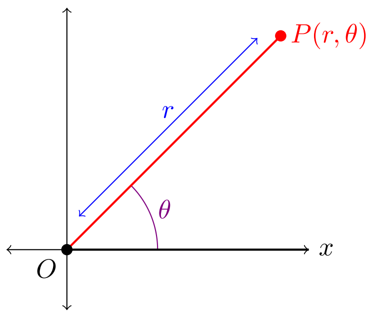
$$\frac{dx}{dt} = -\sin(t)$$

$$\frac{dy}{dt} = \cos(t)$$

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 2\pi \int_0^{2\pi} (2 + \sin(t)) \sqrt{(-\sin(t))^2 + (\cos(t))}\, dt \\ &= 2\pi \int_0^{2\pi} 2 + \sin(t)\, dt \\ &= 2\pi \left[2t - \cos(t)\right]_0^{2\pi} \\ &= 2\pi \left[(4\pi - 1) - (0 - 1)\right] \\ &= \boxed{8\pi^2} \end{aligned}$$

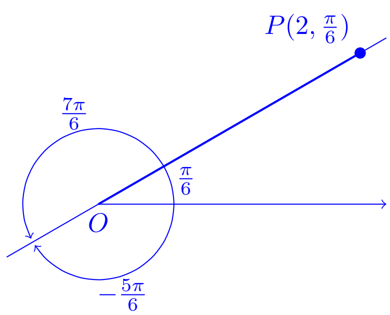
Section 11.3: Polar Coordinates

Definition: To define polar coordinates, we first fix an ____origin____ O (called the ____pole____) and an ____initial ray____ from O (usually the positive x-axis). Then each point P can be located by assigning to it a ____polar coordinate pair____ (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray OP.



Just like trigonometry, θ is positive when measured anticlockwise and negative when measured clockwise. The angle associated with a given point is not unique.
In some cases, we allow r to be negative. For instance, the point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians anticlockwise from the initial ray and going forward 2 units, or we could turn $\pi/6$ radians clockwise and go backwards 2 units; corresponding to $P(-2, \pi/6)$.

Example 1: Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.



For $r = 2$,

$$\theta = \frac{\pi}{6},\ \frac{\pi}{6} \pm 2\pi,\ \frac{\pi}{6} \pm 4\pi,\ \frac{\pi}{6} \pm 6\pi,\ \ldots$$

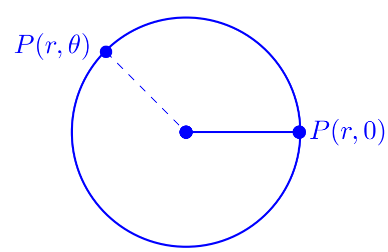
For $r = -2$,

$$\theta = -\frac{5\pi}{6},\ -\frac{5\pi}{6} \pm 2\pi,\ -\frac{5\pi}{6} \pm 4\pi,\ -\frac{5\pi}{6} \pm 6\pi,\ \ldots$$

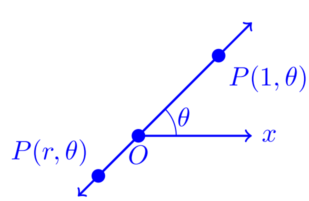
So,

$$\left\{ \left(2, \frac{\pi}{6} + 2n\pi\right), \left(-2, -\frac{5\pi}{6} + 2n\pi\right) \mid n \in \mathbb{Z} \right\}$$

Polar Equations and Graphs: If we fix r at a constant value (not equal to zero), the point $P(r, \theta)$ will lie $|r|$ unites from the origin O. As θ varies over any interval of length 2π, P traces a what? A circle!



If we fix θ at a constant value and let r vary between $-\infty$ and ∞, then the point $P(r, \theta)$ traces a what? A line!



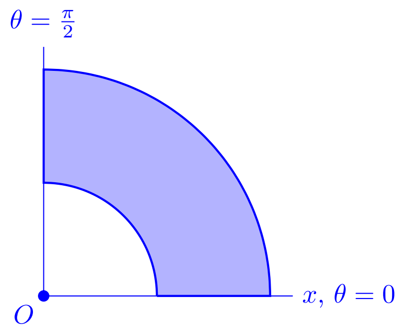
Example 2: A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for a circle of radius 1 centred at the origin.

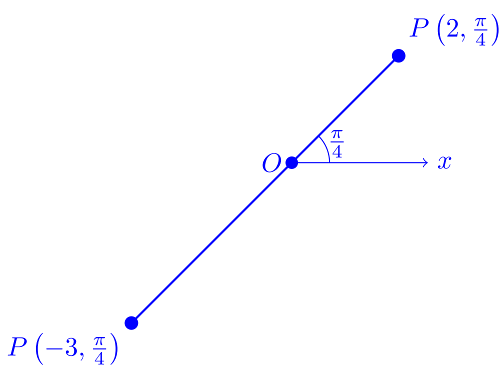
(b) $\theta = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{5\pi}{6}, \ldots$ are all equations for the line passing through the Cartesian points $(0, 0)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Example 3: Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments and rays. Graph the sets of points whose polar coordinates satisfy the given conditions:

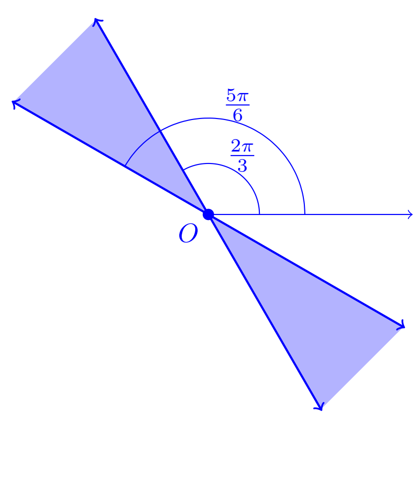
(a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$



(b) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$



(c) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$



Relating Polar and Cartesian Coordinates: When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by the following:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r^2 = x^2 + y^2.$$

Example 4: Given the polar equation, find the Cartesian equivalent:

(a) $r\cos(\theta) = 2$

$$\boxed{x = 2}$$

(b) $r^2\cos(\theta)\sin(\theta) = 4$

$$r\cos(\theta) \cdot r\sin(\theta) = 4 \Longrightarrow \boxed{xy = 4}$$

(c) $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$

$$(r\cos(\theta))^2 - (r\sin(\theta))^2 = 1 \Longrightarrow \boxed{x^2 - y^2 = 1}$$

(d) $r = 1 + 2r\cos(\theta)$

$$r^2 = (1 + 2r\cos(\theta))^2 = 1 + 4r\cos(\theta) + 4\,(r\cos(\theta))^2 \Longrightarrow \boxed{x^2 + y^2 = 1 + 4x + 4x^2}$$

(e) $r = 1 - \cos(\theta)$

$$\begin{aligned} r^2 = (1 - \cos(\theta))\,r = r - r\cos(\theta) &\Longrightarrow r^2 + r\cos(\theta) = r \\ &\Longrightarrow \left(r^2 + r\cos(\theta)\right)^2 = r^2 \\ &\Longrightarrow \boxed{\left(x^2 + y^2 + x\right)^2 = x^2 + y^2} \end{aligned}$$

Example 5: Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$.

$$x^2 + y^2 - 6y + 9 = 9 \Longrightarrow \left(x^2 + y^2\right) - 6y = 0 \Longrightarrow \boxed{r^2 - 6r\sin(\theta) = 0}$$